New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **FALL 2002**

PART I: 10 minutes | **NYCIML Contest One** | **Fall 2002**

F02J1. 1729 can be written as the sum of the cubes of two positive integers ($1729 = a^3 + b^3,\ a < b$) in two ways. Compute the two possible ordered pairs (a,b).

F02J2. A 50-pound sponge is 99% water. After it is left out in the sun, the sponge turns to 97% water. Compute the new weight of the sponge.

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2002**

F02J3. When the four-digit integer represented by $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is multiplied by 9, the result is $\underline{D}\,\underline{C}\,\underline{B}\,\underline{A}$. If A, B, C, and D are all distinct, compute the number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$.

F02J4. Brenda is collecting money from her relatives. She starts with no money, and each relative she goes to gives her either a quarter or a dime, with an equal chance of each. Compute the probability that at some point Brenda will have exactly \$1.00.

PART III: 10 minutes | **NYCIML Contest One** | **Fall 2002**

F02J5. Harry is a seeker, or Hermione is smart. If Harry is a seeker, then Fluffy is dangerous. If Hermione is smart, then Voldermort is evil. If Fluffy is not dangerous, then Voldermort is not evil. Based on the statements, is Fluffy dangerous or not? Or is it impossible to tell?

F02J6. Compute the two possible positive integral values of x such that $x^2 + x + 39$ is a perfect square.

ANSWERS:

F02J1. $(1,12)$ and $(9,10)$

F02J2. $\frac{50}{3}$ pounds

F02J3. 1089

F02J4. $\frac{233}{1024}$

F02J5. Fluffy is dangerous

F02J6. 6 and 38

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2002**

PART I: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02J7. If Ryan gets a 90 on his next test, his average will increase by 1. If he gets a 72, his average will decrease by 5. How many tests has Ryan already taken?

F02J8. Jennifer bought 100 flowers for \$99. There were roses, tulips and daisies. Each rose cost \$6, each tulip \$4 and each daisy cost 50 cents. Compute how many of each flower she bought.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02J9. A student guesses at random the answers to five true-false questions. Compute the probability she guesses at least three questions correctly.

F02J10. Compute the smallest three-digit integer that is the sum of the cubes of its digits.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02J11. The product of two positive integers is 112. If one is divided by the other, the quotient is 2 and the remainder is 2. Compute the numbers.

F02J12. In isosceles triangle GHI, $\overline{GH}$ is the base. Points A and B are chosen on $\overline{GI}$ and $\overline{HI}$, respectively, such that $\overline{AB} \parallel \overline{GH}$, and the segment $\overline{AB}$ divides the area of triangle GHI and the perimeter of triangle GHI into two equal parts. Compute $\cos \angle IGH$.

ANSWERS:

F02J7. 2

F02J8. 7 roses, 3 tulips, and 90 daisies

F02J9. $\frac{1}{2}$

F02J10. 153

F02J11. 16 and 7

F02J12. $\sqrt{2}-1$

JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 2002**

PART I: 10 minutes | **NYCIML Contest Three** | **Fall 2002**

F02J13. Compute the smallest two-digit positive integer that is the sum of the cube of one of its digits and the square of the other digit.

F02J14. If $2x+y=n$ and $x+2y=m$ and $(x+y)^2=2(x-y)^2$, compute $\frac{m}{n}+\frac{n}{m}$.

PART II: 10 minutes | **NYCIML Contest Three** | **Fall 2002**

F02J15. Compute: $12+\sqrt{12+\sqrt{12+\sqrt{12+...}}}$

F02J16. Perpendiculars $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ are drawn from the vertices of parallelogram $ABCD$ to a line that does not intersect the parallelogram. If $AE=46$, $BF=33$ and $CG=84$, compute DH.

PART III: 10 minutes | **NYCIML Contest Three** | **Fall 2002**

F02J17. Define the function $g(x)=x\,g(-1)$ for all real values of x. Compute $g(2003)$.

F02J18. Compute the number of ordered pairs of integers (x, y) such that $\sqrt{5103}=\sqrt{x}+\sqrt{y}$.

ANSWERS:

F02J13. 24
F02J14. $\frac{38}{17}$
F02J15. 16
F02J16. 97
F02J17. 0
F02J18. 28

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JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2002**

Solutions

F02J1. $2(9^3) = 1458 < 1729 < 2197 = 13^3$, so $9 < b < 13$. Checking the three values we find that the only solutions are $\mathbf{(1,12)}$ **and** $\mathbf{(9,10)}$.

F02J2. The non-water part of the sponge weighs 0.5 pounds. If this part is 3% of the weight of the dried sponge, the new weight of the sponge is $\mathbf{\frac{50}{3}}$ **pounds**.

F02J3. $A = 1$, since $D \le 9$. By the same logic, $B = 0$, and so $D = 9$. $\underline{D}\,\underline{C}\,\underline{B}\,\underline{A}$ is divisible by 9, so $A + B + C + D$ is also divisible by 9, and $C = 8$. We can check that in fact **1089** is the correct answer.

F02J4. Brenda can get to \$1.00 in one of three ways.

(i) The first four relatives will each give her a quarter. $P(i) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

(ii) The first ten relatives will each give her a dime. $P(ii) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$

(iii) The first seven relatives give her a combination of two quarters and five dimes.

$$P(iii) = \binom{7}{2}\left(\frac{1}{2}\right)^7 = \frac{21}{128}$$

These cases are exclusive and account for all possibilities, so

$$P = P(i) + P(ii) + P(iii) = \frac{1}{16} + \frac{1}{1024} + \frac{21}{128} = \mathbf{\frac{233}{1024}}$$

F02J5. If Fluffy is not dangerous, then Voldemort is not evil. We can take the contrapositive of that statement and get If Voldemort is evil then Fluffy is dangerous. Therefore following from another given we get: If Hermione is smart then Fluffy must be dangerous. Therefore, now we know that Fluffy must be dangerous.

F02J6. Let $y^2 = x^2 + x + 39$. $4y^2 = 4x^2 + 4x + 156 = (2x+1)^2 + 155$, so $155 = (2y)^2 - (2x+1)^2 = (2y + 2x + 1)(2y - 2x - 1)$. Since the first factor is larger than the second factor, and 155 can only be expressed as a product as 155×1 or 31×5, we have only those two cases to consider. Solving for x we obtain **6 and 38** as the only solutions.

New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2002**

Solutions

F02J7. Let a denote Ryan's current average and n denote the number of tests he has taken. $\frac{an+90}{n+1}=a+1$ and $\frac{an+72}{n+1}=a-5$. Solving for n we find $\boldsymbol{n=2}$.

F02J8. Let r, t, and d be the numbers of roses, tulips, and daisies, respectively. $r+t+d=100$ and $6r+4t+\frac{d}{2}=99$, so $4r+602=7d$, so 7 divides r. 21 or more roses would cost more than \$99. 14 roses would cost \$84, and the remaining 86 flowers would cost more than \$15. Thus there were 7 roses. Solving $t+d=93$ and $4t+\frac{d}{2}=57$ we find that Jennifer bought **7 roses, 3 tulips, and 90 daisies.**

F02J9. The student will get either at least 3 questions correct or at least 3 questions wrong. Since the occurrences are symmetrical, their probabilities are equal and both $\boldsymbol{\frac{1}{2}}$.

F02J10. Since we are looking for the smallest integer, we should first try one with the first digit 1. $6^3=216>199$ and $3^3=27<\frac{100}{2}$, so one of the other digits will have to be 4 or 5. Trying all possible cases we find that $1^3+5^3+3^3=\mathbf{153}$ is the smallest solution.

F02J11. $ab=112$ and $2a+2=b$, so $a(2a+2)=112$ or $a^2+a-56=0$. The positive solution is $a=7$ and the two numbers are **7 and 16**.

F02J12. Let $GI=HI=y$ and $GH=2x$ (so $\cos\angle IGH=\frac{x}{y}$). $\Delta GHI \sim \Delta ABI$ and the ratio of their areas is 2:1, so the ratio of their respective sides is $\sqrt{2}:1$. Thus $AI=BI=\frac{y\sqrt{2}}{2}$. We have $2(AI+BI)=GH+GI+HI$, so $2\left(\frac{y\sqrt{2}}{2}+\frac{y\sqrt{2}}{2}\right)=2x+2y$, and $\cos\angle IGH=\frac{x}{y}=\frac{2\sqrt{2}-2}{2}=\boldsymbol{\sqrt{2}-1}$.

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2002**

Solutions

F02J13. If the number starts with 1 then it would have to equal either $3^2+1=10$ or $4^2+1=17$, both impossible. If the number starts with 2 then it would have to equal $4^2+2^3=\mathbf{24}$, which is the smallest solution.

F02J14. $x+y=\frac{n+m}{3}$, $x-y=n-m$, so $\left(\frac{n+m}{3}\right)^2=2(n-m)^2$, or $17n^2+17m^2=38mn$, so $\frac{m}{n}+\frac{n}{m}=\frac{m^2+n^2}{mn}=\mathbf{\frac{38}{17}}$

F02J15. Let $x=12+\sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$. We see that the infinite sum is also equal to $12+\sqrt{x}$. Hence we only have to solve the equation $x=12+\sqrt{x}$. Solving the equation, we find that x is equal to 16.

F02J16. Because $ABCD$ is a parallelogram we will have $AE-BF=DH-CG$. Thus $DH=AE+CG-BF=\mathbf{97}$

F02J17. We take $g(-1)$: $g(-1)=-1(g(-1))$, so $2(g(-1))=0$, and $g(-1)=0$. Hence $g(x)=0$ for all x, and $g(2003)=0$

F02J18 $\sqrt{5103}=\sqrt{x}+\sqrt{y}\rightarrow y=5103+x-2\sqrt{5103x}$. It follows that y is an integer iff $5103x$ is a perfect square. $5103=3^6\cdot 7$, hence x must be of the form $7z^2$, moreover we know $0\le x\le 5103$, thus $0\le z\le 27$. Therefore there are **28** such ordered pairs.