New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER ONE** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest One** **Spring 2002**

S02B1. Three faces of a rectangular solid have areas 12, 15 and 20. Compute the volume of the solid.

S02B2. If $x = \frac{a}{b}$, express $\frac{a+b}{a-b}$ in terms of x as a fraction in simplest form.

PART II: 10 minutes **NYCIML Contest One** **Spring 2002**

S02B3. Given square $ABCD$, N is the midpoint of $\overline{AB}$ and M is the midpoint of $\overline{BC}$. Compute the $\sin(\measuredangle MDN)$.

S02B4. If x and y are integers, solve for all values of (x, y): $2^{2x} - 3^{2y} = 55$.

PART III: 10 minutes **NYCIML Contest One** **Spring 2002**

S02B5. Compute the remainder when $x^6 + x^4 + 1$ is divided by x-1.

S02B6. Compute the number of integral solutions of $(x^2 - 3x + 1)^{x^2+7x+12} = 1$.

ANSWERS

1. 60
2. $\frac{x+1}{x-1}$
3. $\frac{3}{5}$
4. $(3,1)$
5. 3
6. 6

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SENIOR B DIVISION **CONTEST NUMBER TWO** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02B7. If $\sqrt{x\sqrt{x}} = x^y$ and $x > 0$, compute y.

S02B8. Two circles are concentric. A chord of the larger is tangent to the smaller. If the length of this chord is 12, compute the area of the region between the circles.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02B9. If x and y are positive integers, compute the number of solutions (x, y) for $5x + 3y = 2001$.

S02B10. If $[x]$ represents the greatest integer less then or equal to x, compute the smallest value of x for which $[x] + [2x] + [3x] + [4x] = 16$.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02B11. The lengths of the sides of a triangle with area x are 2, 7 and x. Compute x.

S02B12. The sum of an infinite geometric progression is 6 and the sum of the first 2 terms is $4\frac{1}{2}$. Compute all possible values of the first term.

ANSWERS

7. $\frac{3}{4}$
8. 36π
9. 133
10. $1\frac{3}{4}$ or $\frac{7}{4}$
11. $3\sqrt{5}$
12. 3,9

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SENIOR B DIVISION **CONTEST NUMBER THREE** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02B13. Solve for x : $4^x - 4^{x-1} = 24$.

S02B14. Point P is 12 units from the center of a circle with radius 20. How many chords with different integral lengths can be drawn through P?

PART II: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02B15. If $\log_a b \cdot \log_5 a = 4$, compute b.

S02B16. Compute the value of $\frac{\sqrt{7}}{3+\sqrt{8}}+\frac{\sqrt{7}}{\sqrt{8}+\sqrt{7}}+\frac{\sqrt{7}}{\sqrt{7}+\sqrt{6}}+\frac{\sqrt{7}}{\sqrt{6}+\sqrt{5}}+\frac{\sqrt{7}}{\sqrt{5}+2}$.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02B17. Compute $(1-i)^{20}$.

S02B18. Compute the number of times in a day that the hands of a clock are perpendicular to each other.

ANSWERS

13. $\frac{5}{2}$
14. **9**
15. **625**
16. $\sqrt{7}$
17. **-1024**
18. **44**

SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2002**

S02B19. If the altitude of an equilateral triangle is $3\sqrt{2}$, compute the area of the triangle.

S02B20. If the same number is added to 5, 15, and 30, the three resulting numbers form a geometric progression. Compute the number.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2002**

S02B21. Solve for x: $\sqrt{5-x} = x\sqrt{5-x}$.

S02B22. How many non-congruent scalene triangles can be constructed with sides of integral lengths and perimeter less than 15?

PART III: 10 minutes **NYCIML Contest Four** **Spring 2002**

S02B23. Compute the sum of the reciprocals of the roots of $x^2 - 11x + 9 = 0$.

S02B24. Compute the probability that this 14 digit number 3A,5B6,C57,8D6,E44 is divisible by 396 where A, B, C, D, E corresponds to 1, 2, 3, 4, 5 (though not necessarily in that order).

ANSWERS

19. $6\sqrt{3}$
20. **15**
21. **5,1**
22. **6**
23. $\frac{11}{9}$
24. **1**

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SENIOR B DIVISION | **CONTEST NUMBER FIVE** | **SPRING 2002**

PART I: 10 minutes | **NYCIML Contest Five** | **Spring 2002**

S02B25. Compute $\log_2\left(16\cdot\sqrt[4]{8}\sqrt[3]{32}\right)$.

S02B26. The radii of 2 circles are 6 and 8 and the distance between their centers is 20. Compute the length of a common internal tangent segment.

PART II: 10 minutes | **NYCIML Contest Five** | **Spring 2002**

S02B27. Compute x: $9^{x-2} = 9^{x} + 240$.

S02B28. A subset of the first 100 positive integers consists of integers, none of which is twice another integer in the set. What is the maximum number of integers that can be in this subset?

PART III: 10 minutes | **NYCIML Contest Five** | **Spring 2002**

S02B29. A triangle is inscribed in a circle. If one side is 8 and the angle opposite this side is 30°, compute the area of the circle.

S02B30. 15 billiard balls, numbered 1 to 15 are on the table. If 5 are pocketed, compute the probability that their sum is odd. (Reminder: fractions must be reduced to lowest terms).

ANSWERS

25. $\frac{77}{12}$
26. $2\sqrt{51}$
27. $\frac{1}{2}$
28. 67
29. 64π
30. $\frac{72}{143}$

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SENIOR B DIVISION **CONTEST NUMBER ONE** **Spring 2002**

SOLUTIONS

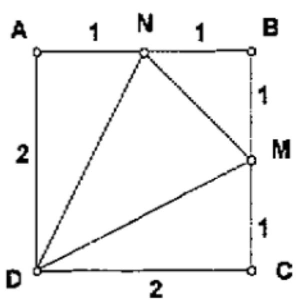
S02B1. $\boxed{60}$

Let x, y and z be the lengths of the edges. $xy = 12$, $yz = 15$, $xz = 20$.
Multiplying $x^2y^2z^2 = 3600$, $xyz = 60$. V = xyz = 60.

S02B2. $\boxed{\dfrac{x+1}{x-1}}$

In $\dfrac{a+b}{a-b}$ divide numerator and denominator by b. $\dfrac{\frac{a}{b}+1}{\frac{a}{b}-1} = \dfrac{x+1}{x-1}$.

S02B3. $\boxed{\dfrac{3}{5}}$



Let the side of the square equal 2. $MD = DN = \sqrt{5}$.
Area of $\Delta MDN = \frac{1}{2}\sqrt{5}\sqrt{5}\sin(\sphericalangle MDN)$. Area of
$\Delta MDN = \frac{3}{2} = \frac{5}{2}\sin(\sphericalangle MDN)$. $\frac{3}{5} = \sin(\sphericalangle MDN)$.

S02B4. $\boxed{(3,1)}$

Factoring, $\left(2^x + 3^y\right)\left(2^x - 3^y\right) = 55$. The only integral pairs of factors of 55 are 55×1 and 11×5. However only 11×5 will produce integral solutions
$2^x + 3^y = 11 \quad 2^x - 3^y = 5 \quad 2\cdot 2^x = 16 \quad 2^x = 8$, $x = 3$ $y = 1$.

S02B5. $\boxed{3}$

This can be done using long division. However, the remainder theorem is easier. Remainder when $f(x)$ is divided by $x-a$ is $f(a)$.
$f(1) = 1+1+1 = 3$.

S02B6. $\boxed{6}$

There are 3 possibilities

a) $x^2+7x+12=0$ and $x^2-3x+1 \neq 0 \quad x=-3,-4$

b) $x^2-3x+1=1 \quad x=0,3$

c) $x^2-3x+1=-1$ and $x^2+7x+12$ is even
$x^2-3x+2=0$
$x=1,2$ and both make $x^2+7x+12$ even.

New York City Interscholastic Mathematics League

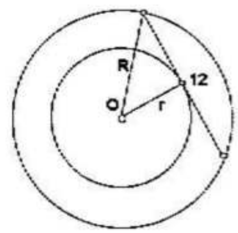
SENIOR B DIVISION **CONTEST NUMBER TWO** **Spring 2002**

SOLUTIONS

S02B7. $\boxed{\frac{3}{4}}$

$\sqrt{x\sqrt{x}} = \sqrt{x^{\frac{3}{2}}} = x^{\frac{3}{4}}$.

S02B8. $\boxed{36\pi}$



$R^2 - r^2 = 6^2 = 36$

$\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right) = 36\pi$.

S02B9. $\boxed{133}$

$5x + 3y = 2001$, $y = \dfrac{2001-5x}{3}$. Since 2001 is a multiple of 3, x must be as well. x can be any multiple of 3 from 3 to 399. 133 possibilities.

S02B10. $\boxed{1\frac{3}{4}}$

Let $f(x) = [x] + [2x] + [3x] + [4x]$. $f(1) = 10$, $f(1\frac{1}{2}) = 14$, $f(1\frac{1}{3}) = 12$, $f(1\frac{1}{4}) = 11$, $f(1\frac{2}{3}) = 15$, $f(1\frac{3}{4}) = 16$.

S02B11. $\boxed{3\sqrt{5}}$

Let A be the angle between the sides of length 2 and x. $x = \frac{1}{2}\cdot 2\cdot x\cdot \sin A$.
Sin A =1 and therefore A is a right angle. The right triangle has sides 2, $3\sqrt{5}$, and 7 with area $3\sqrt{5}$.

S02B12. $\boxed{3,9}$

$\dfrac{a}{1-r} = 6 \Rightarrow a = 6(1-r)$

$a + ar = \dfrac{9}{2} \Rightarrow 6(1-r) + 6r(1-r) = \dfrac{9}{2}$. Factoring, $6(1+r)(1-r) = \dfrac{9}{2}$,

$1 - r^2 = \dfrac{9}{12} = \dfrac{3}{4}$, $r^2 = \dfrac{1}{4} \Rightarrow r = \pm\dfrac{1}{2}$. If $r = \dfrac{1}{2}$, $a = 3$; if $r = \dfrac{-1}{2}$, $a = 9$.

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SENIOR B DIVISION **CONTEST NUMBER THREE** **Spring 2002**

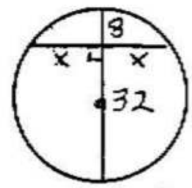
SOLUTIONS

S02B13. $\boxed{\frac{5}{2}}$

$4^{x-1}(4-1)=24$

$3\left(4^{x-1}\right)=24$

$4^{x-1}=8;\ x-1=\frac{3}{2},\ x=\frac{5}{2}.$



$x^2 = 8 \cdot 32$

$x = 16$

$2x = 32$

S02B14. $\boxed{9}$

The shortest chord that can be drawn has length 32, as shown. The longest is a diameter of length 40. All integral lengths between these two numbers are possible⇒ nine possibilities.

S02B15. $\boxed{625}$

Let $x=\log_a b,\ \ y=\log_5 a$.

$a^x=b,\ 5^y=a,\ \left(5^y\right)^x=b,\ 5^{xy}=b.$

$5^4=b,\ b=625.$

or

Notice a is NOT specified. Therefore any a will work.
Try a convenient a: $a=5$.
$\log_5 b\cdot\log_5 5=4$

$\log_5 b=4,\ b=5^4.$

S02B16. $\boxed{\sqrt{7}}$

Rationalizing all the denominators, they all become 1.
$\sqrt{7}\left(3-\sqrt{8}+\sqrt{8}-\sqrt{7}+\sqrt{7}-\sqrt{6}+\sqrt{6}-\sqrt{5}+\sqrt{5}-2\right)=\sqrt{7}.$

S02B17. $\boxed{-1024}$

$(1-i)^2=1-2i+i^2=-2i$

$(1-i)^{20}=(-2i)^{10}=1024i^{10}=-1024.$

S02B18. $\boxed{44}$

At first glance, it would seem as if this occurs twice every hour. However, since they are perpendicular at 3 o'clock and at 9 o'clock there are only three times between 2 and 4, and 3 times between 8 and 10. There are 44 times.

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **Spring 2002**

SOLUTIONS

S02B19. $\boxed{6\sqrt{3}}$

If the altitude is $3\sqrt{2}$, the base $= 2\dfrac{\sqrt{18}}{\sqrt{3}} = 2\sqrt{6}$.

$A = \dfrac{\sqrt{3}}{4}s^2 = \dfrac{\sqrt{3}}{4}\left(2\sqrt{6}\right)^2 = 6\sqrt{3}$.

S02B20. $\boxed{15}$

$$\frac{x+5}{x+15} = \frac{x+15}{x+30}.$$
$$x^2+30x+225 = x^2+35x+150$$
$$75 = 5x$$
$$15 = x.$$

S02B21. $\boxed{5,1}$

Squaring both sides $5-x = x^2(5-x)$. $(5-x)(x^2-1)=0$. Possible solutions are 5, ± 1. However, -1 does not check in the original equation.

S02B22. $\boxed{6}$

1 cannot be a side, since the largest side cannot be equal to the sum of the other two sides. Starting with 2: 2, 3, 4; 2, 4, 5; 2, 5, 6; 3, 4, 5; 3, 4, 6; 3, 5, 6.

S02B23. $\boxed{\dfrac{11}{9}}$

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1+R_2}{R_1R_2} = \frac{11}{9}.$$

S02B24. $\boxed{1}$

$396 = 4\cdot 9\cdot 11$ and since these numbers are relatively prime, it is sufficient to check divisibility by these 3 numbers.

a) The number is divisible by 4 since the last two digits are 44.
b) The number is divisible by 9 since the sum of the digits is 63, regardless of where the numbers are placed.
c) The sum of the odd placed digits is 37 and the even placed digits is 26, regardless of where the numbers are placed. Since this difference is 11, the number is divisible by 11. The number MUST be divisible by 396.

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **Spring 2002**

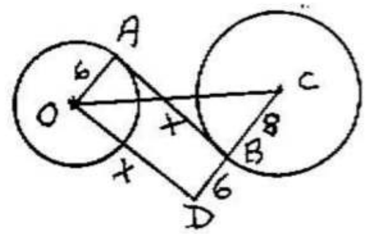
SOLUTIONS

S02B25. $\boxed{\dfrac{77}{12}}$

$$\log_2\left(2^4 2^{\frac{3}{4}} 2^{\frac{5}{3}}\right)=\log_2\left(2^{\frac{77}{12}}\right)=\frac{77}{12}.$$

S02B26. $\boxed{2\sqrt{51}}$

Extend $\overline{CB}$ a length of 6. $OABD$ is a rectangle. $OD=AB$. In ΔODC, $14^2+x^2=20^2$, $x^2=204$, $x=\sqrt{204}=2\sqrt{51}$.



S02B27. $\boxed{\dfrac{1}{2}}$

$$\begin{aligned} 9^x\left(9^2-1\right)&=240\\ 80\cdot 9^x&=240\\ 9^x&=3\\ x&=\frac{1}{2}. \end{aligned}$$

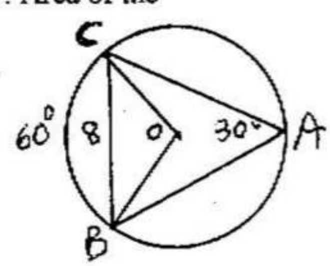
S02B28. $\boxed{67}$

Obviously, all odd numbers can be included. Then, multiples of 4 which are not multiples of 8 could be included: 4, 12. 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100. Finally, 4 multiples of 16 can be included – 16, 48, 64, and 80. 67 numbers.

S02B29. $\boxed{64\pi}$

Draw $\overline{OB}$ and $\overline{OC}$. $m\angle BOC=60^0$, ΔBOC is equilateral. $BO=8$. Area of the circle = 64π.

Note: This will work whether or not the center is inside the angle.



S02B30. $\boxed{\dfrac{72}{143}}$

Either, all 5 are odd, 3 are odd, or 1 is odd.

$$\frac{{}_8C_5+{}_8C_3\cdot{}_7C_2+{}_8C_1\cdot{}_7C_4}{{}_{15}C_5}=\frac{56+1176+280}{3003}=\frac{1512}{3003}=\frac{72}{143}.$$