NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION

CONTEST NUMBER ONE

PART I	*SPRING, 2002*	*CONTEST 1*	*TIME: 10 MINUTES*

S02S1 If $x^2 + y^2 = 2002$ and $\frac{1}{x^2} + \frac{1}{y^2} = 2002$, compute all possible values of xy.

S02S2 A quarter has a one-inch diameter. The quarter is rolled along the perimeter of an equilateral triangle with a side of length 2 inches. The quarter always lies on the exterior of the triangle. Compute the area, in square inches, covered by moving the quarter around the entire triangle.

PART II	*SPRING, 2002*	*CONTEST 1*	*TIME: 10 MINUTES*

S02S3 The Smarandache function is defined $S(n) = k$, where n is any positive integer and k is the smallest positive integer greater than 1 such that n divides $k!$. Compute k such that $S(2000) = k$.

S02S4 $\sum_{k=13}^{2002} \left[\log\left(k^2 - 9\right) - \log\left(k^2 - 4\right) \right] = \log\left(\frac{a}{b}\right)$ where a and b are relatively prime integers. Compute $a + b$.

PART III	*SPRING, 2002*	*CONTEST 1*	*TIME: 10 MINUTES*

S02S5 Harry hits p % of his foul shots, worth one point each. The Wizards are losing by one, when Harry shoots a foul shot for the Wizards. If he hits it, Harry will get a second shot. If he misses it, he does not get a second shot. The probability that the Wizards are ahead after Harry's shot(s) is equal to the probability that they are still losing. Compute p.

S02S6 When written in base 5, a positive integer B has two terminal zeroes. When written in base 2, B has three terminal zeroes. In base 3, B has one terminal zero. In how many other positive integral bases greater than 1 must the representation of B have at least one terminal zero?

ANSWERS:

S02S1	± 1
S02S2	$(6 + \pi)$ square inches
S02S3	15
S02S4	1001
S02S5	$50\sqrt{5} - 50$
S02S6	20

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CONTEST NUMBER TWO

PART I	*SPRING, 2002*	*CONTEST 2*	*TIME: 10 MINUTES*

S02S7 Compute the number of digits in the decimal numeral $2^{2002} \times 5^{2005}$.

S02S8 The game of Parabolum uses the equation $x^2 + ax - b = 0$. Player A rolls a fair four-sided die with the numbers 0, 1, 2, and 3, to choose the value of a. Player B rolls a fair ten-sided die with the integers zero to nine inclusive to determine the value of b. Player B wins if the roots are rational otherwise Player A wins. Compute the probability that Player B will win.

PART II	*SPRING, 2002*	*CONTEST 2*	*TIME: 10 MINUTES*

S02S9 Compute all x such that: $\left|\frac{x-1}{x+1}\right| + |x+2| = 3$.

S02S10 The perimeter of triangle *NYC* is 26. The altitudes of triangle *NYC* are in the ratio 2 : 3 : 4. Compute the area of triangle *NYC*.

PART III	*SPRING, 2002*	*CONTEST 2*	*TIME: 10 MINUTES*

S02S11 $2^{-(2x-1)} - 2^{-(2x+1)} - 2^{-2x} = 2^a$. Express a in simplest form in terms of x.

S02S12 The following four lines are tangent to circle O;

$x + 2y = a$, $x + 2y = b$,

$x + y = 9$, $x + y = 17$.

If $a \neq b$, compute $|a - b|$.

ANSWERS:		
	S02S7	2005
	S02S8	$\frac{3}{10}$
	S02S9	0, 1, -2, -3
	S02S10	$\sqrt{455}$
	S02S11	$-2x-1$
	S02S12	$4\sqrt{10}$

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CONTEST NUMBER THREE

PART I ***SPRING, 2002*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S02S13 Compute the largest prime factor of $5^8 + 5^6 - 5^2 - 5^0$.

S02S14 Three bags contain marbles. The first bag contains two white and one black marble. The second contains one white and two black marbles. And the last bag has six black marbles. A bag is chosen at random and then a marble is selected. If the chosen marble is black, compute the probability that the bag contains all black marbles.

PART II ***SPRING, 2002*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S02S15 Compute the value of: $\dfrac{1}{\sqrt{2+\dfrac{1}{\sqrt{2+\dfrac{1}{\sqrt{2+\sqrt{\dfrac{1}{\ldots}}}}}}}}$

S02S16 A triangle has sides of length 21, 28, and 35. Circle O is drawn with its center on the longest side of the triangle. Circle O is tangent to the other two sides. Compute the length of the radius of circle O.

PART III ***SPRING, 2002*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S02S17 Compute all possible values for x.
$64 + 64x^2 + 64x^4 + 64x^6 + \ldots = 100$

S02S18 A quarter has a one-inch diameter. The quarter is rolled around the interior of an equilateral triangle with a side of length two inches such that the quarter is tangent to at least one side of the triangle. The quarter never leaves the interior of the triangle. Compute the area, in square inches, covered by moving the quarter in the triangle.

ANSWERS:

S02S13	31
S02S14	½
S02S15	$\frac{\sqrt{5}-1}{2}$
S02S16	12
S02S17	$\pm\frac{3}{5}$
S02S18	$\frac{\sqrt{3}}{4}+\frac{\pi}{4}$

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CONTEST NUMBER FOUR

PART I	***SPRING, 2002***	***CONTEST 4***	***TIME: 10 MINUTES***

S02S19 Compute the number of integers between 1 and 2002, inclusive, which contain at least one digit that is a 0, 1 or 2.

S02S20 Three circles are mutually externally tangent. Each circle is tangent to line l. Two of the circles are congruent, with radius r. The other circle has a radius of 5. Compute r.

PART II	***SPRING, 2002***	***CONTEST 4***	***TIME: 10 MINUTES***

S02S21 Compute all x such that: $(64x^2-36)^3+(36x^2-64)^3=(100x^2-100)^3$.

S02S22 The equation $9(2^{x+1})+2^{1-x}=a$ has a single real root for x. Compute all possible values of a.

PART III	***SPRING, 2002***	***CONTEST 4***	***TIME: 10 MINUTES***

S02S23 Compute $\sum_{n=1}^{2002}\sin\left(\frac{n\pi}{6}\right)\cos\left(\frac{n\pi}{6}\right)$.

S02S24 A triangle has three altitudes of length 5, 4, and y. If $a<y<b$, compute $\frac{b}{a}$.

ANSWERS:

S02S19	1603
S02S20	20
S02S21	$1, \pm\frac{3}{4}, \pm\frac{4}{3}$
S02S22	12
S02S23	$\frac{\sqrt{3}}{4}$
S02S24	9

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CONTEST NUMBER FIVE

PART I *SPRING, 2002* *CONTEST 5* *TIME: 10 MINUTES*

S02S25 If $x^3-2x^2+20x-12=A(x-2)^3+B(x-2)^2+C(x-2)+D$, compute $A + C$.

S02S26 Compute the smallest positive integer x greater than 10 such that $[x]-24\left[\frac{x}{24}\right]=10$ and $[x]-83\left[\frac{x}{83}\right]=10$. ([x] is the greatest integer less than or equal to x)

PART II *SPRING, 2002* *CONTEST 5* *TIME: 10 MINUTES*

S02S27 $2-\sin A=\sqrt{2\cos^2 A+\sin A}$, compute all possible values for $\sin 2A$.

S02S28 Compute:

$$\frac{1}{\sqrt{1^3}}+\frac{1}{\sqrt{1^3+2^3}}+\frac{1}{\sqrt{1^3+2^3+3^3}}+\ldots+\frac{1}{\sqrt{1^3+2^3+3^3+\ldots+2002^3}}$$

PART III *SPRING, 2002* *CONTEST 5* *TIME: 10 MINUTES*

S02S29 $1+2+2+3+3+3+\ldots+n+n+\ldots+n=an^3+bn^2+cn$, where the k^{th} natural number appears k times. Compute $a + b + c$.

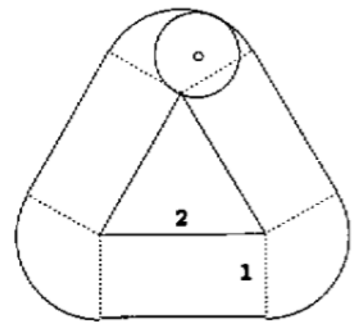
S02S30 $\log_{13}\left(\log_{11}\left(\log_7\left(\log_2 a\right)\right)\right)=0$ Compute the units digit of a.

ANSWERS:

S02S25	25
S02S26	2002
S02S27	$0, \frac{\pm 4\sqrt{5}}{9}$
S02S28	$\frac{4004}{2003}$
S02S29	1
S02S30	8

S02S1 When we divide the first equation by the second, we get $x^2y^2 = 1$, so $xy = \pm\mathbf{1}$.

S02S2 The area the circle covers is composed of three rectangles and three sectors. Each of the rectangles has area 2 and the three sectors can be fit together to form a circle with radius 1 (thus area π). Therefore the total area is $\mathbf{6+\pi}$ square inches.



S02S3 $k!$ has to be divisible by 2^4 and by 5^3. So $k \ge 6$ and $k \ge 15$. The smallest value that works is **15**.

S02S4 $\sum_{k=13}^{2002}\left[\log\left(k^2-9\right)-\log\left(k^2-4\right)\right] = \sum_{k=13}^{2002}\log\frac{(k+3)(k-3)}{(k+2)(k-2)} = \log\prod_{k=13}^{2002}\frac{(k+3)(k-3)}{(k+2)(k-2)}$, so

$$\frac{a}{b} = \frac{(16)(10)}{(15)(11)}\cdot\frac{(17)(11)}{(16)(12)}\cdot\frac{18\cdot 12}{17\cdot 13}\cdot\ldots\cdot\frac{2005\cdot 1999}{2004\cdot 2000} = \frac{10\cdot 2005}{15\cdot 2000} = \frac{401}{600}.\quad a+b=\mathbf{1001}.$$

S02S5 $\left(\frac{p}{100}\right)^2 = 1-\frac{p}{100}$. Solving for the positive value of p we get $p = \mathbf{50\sqrt{5}-50}$.

S02S6 5^2, 2^3, and 3 are all factors of B. Therefore B has at least 24 factors (the number of factors of $5^2\cdot 2^3\cdot 3$). So, other than 1, 2, 3, and 5, B has at least 20 other factors, which correspond to **20** positive integral bases in which B has a terminal 0.

S02S7 $2^{2002}\times 5^{2005} = 10^{2002}\times 5^3 = 10^{2002}\times 125$. This is 1 followed by 2002 zeroes times 125, giving **2005** digits.

S02S8 B wins if (a,b) takes one of the values $(0,0)$, $(0,1)$, $(0,4)$, $(0,9)$, $(1,0)$, $(1,2)$, $(1,6)$, $(2,0)$, $(2,3)$, $(2,8)$, $(3,0)$, or $(3,4)$. The probability that B wins is $\frac{12}{40}=\mathbf{\frac{3}{10}}$.

S02S9 $\left|\frac{x-1}{x+1}\right| = \frac{|x-1|}{|x+1|}$. Solving the equation $\frac{|x-1|}{|x+1|}+|x+2|=3$ in each of the intervals $(-\infty,-2)$, $[-2,-1)$, $[-1,1)$, and $[1,\infty)$, we get the answers **−3, -2, 0, and 1**.

S02S10 Since the product of a side and the altitude drawn to it is the same for all three sides, if the altitudes are in a ratio of 2:3:4, then the sides will be in a ratio 3:4:6, and have lengths 6, 8, and 12. Using Heron's formula, we find the triangle has area $\sqrt{(13)(7)(5)(1)}=\mathbf{\sqrt{455}}$.

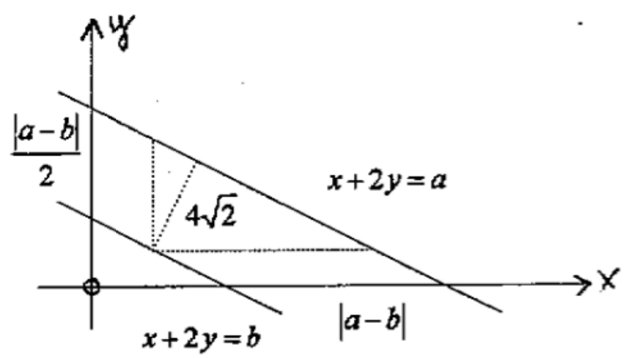
S02S11

$$2^a = 2^{-2x}\cdot 2 - 2^{-2x}\cdot 2^{-1} - 2^{-2x}$$

$$2^a = 2^{-2x}\left(2-\frac{1}{2}-1\right) = 2^{-2x}\cdot 2^{-1} = 2^{-2x-1}$$

$$a = -2x - 1$$

S02S12 The two parallel lines $x+y=9$ and $x+y=17$ are both tangent to circle O, so $4\sqrt{2}$, the distance between the lines, must be the diameter of circle O. Since the lines $x+2y=a$ and $x+2y=b$ are also parallel, the distance between them must also be $4\sqrt{2}$. The horizontal distance between the lines is $|a-b|$, and the vertical distance is $\frac{|a-b|}{2}$. Setting these as the legs of a right triangle, the altitude to the hypotenuse equals $\frac{1}{\sqrt{5}}|a-b| = 4\sqrt{2}$, so



$|a-b| = \mathbf{4\sqrt{10}}$

S02S13 $5^8+5^6-5^2-5^0=$
$(5^2+1)(5^6-1)=26(5^3-1)(5^3+1)=26(124)(126)=2^4\cdot3^2\cdot7\cdot13\cdot31$

S02S14

$$\text{Probability(all black }|\text{black marble appeared)}=\frac{\text{Probability(all black and a black marble appeared)}}{\text{Probability(a black marble appeared)}}$$

$$\text{Probability(a black marble appeared)}=\frac{1}{3}\left(\frac{1}{3}+\frac{2}{3}+\frac{6}{6}\right)=\frac{6}{9}$$

$$\text{Probability(all black and a black marble appeared)}=\frac{1}{3}(1)=\frac{1}{3}$$

$$\text{Probability(all black }|\text{black marble appeared)}=\frac{\frac{1}{3}}{\frac{6}{9}}=\frac{1}{2}$$

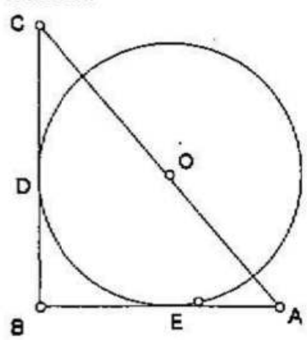
S02S15 Call the expression x. Then,

$$x=\frac{1}{\sqrt{2+x}};x^2=\frac{1}{2+x};x^3+2x^2-1=0$$

$$(x+1)(x^2+x-1)=0;x=-1,\frac{-1+\sqrt{5}}{2},\frac{-1-\sqrt{5}}{2}.$$

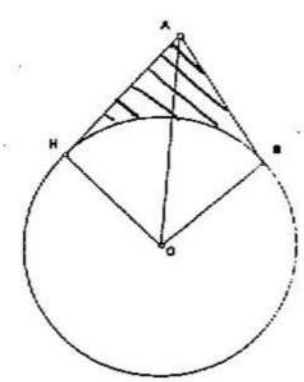
Since $0<x<1$, $x=\dfrac{\sqrt{5}-1}{2}$

S02S16



$\sphericalangle B$ is a right angle (Pythagorean triple 21,28,35) therefore $OD = OE = BE = BD = r$, $AB = 21$.
Using similar triangles, $AE=\dfrac{3r}{4}$ and $AB=\dfrac{7r}{4}=21$ so $r=12$.

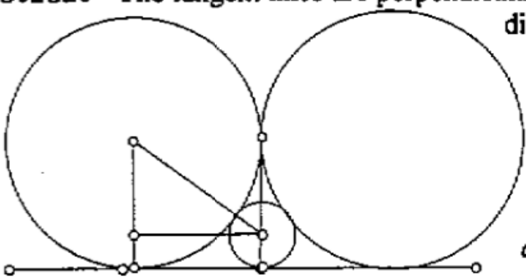
S02S17 We use the sum of an infinite geometric series $S=\dfrac{a}{1-r}$. $S = 100$ $a = 64$ and $r=x^2$. $100=\left(\dfrac{64}{1-x^2}\right)\therefore x=\pm\dfrac{3}{5}$



S02S18 The quarter covers the entire interior of the triangle except the three corners. One is shown in the picture. The area of the shaded region is twice the area of the triangle OHA minus the area of the 120° sector. For all three corners, the area is 6 times the area of triangle OHA minus the area of the quarter. Take the area of the shaded regions from the total area and we get $\dfrac{\sqrt{3}}{4}+\dfrac{\pi}{4}$.

S02S19 To find how many numbers contain at least one 0,1 or 2. It is easiest to count those that have none and subtract it from 2002. If we exclude 0, 1, and 2 there are only seven digits. There are 399 such numbers (7 one-digit numbers, 49 two-digit numbers and 343 three digit numbers). 2002 – 399 = **1603**.

S02S20 The tangent lines are perpendicular to the circles and we can draw a right triangle, as shown in the diagram. The hypotenuse is the sum of both circle's radii, $r + 5$. One leg is the difference between the radii's lengths, $r - 5$. Therefore $(r+5)^2 = r^2 + (r-5)^2$ yields $r = 0$ or $r = 20$.



S02S21 The equation can be written as $a^3 + b^3 = (a+b)^3$, so $3ab(a+b) = 0$. Either $a = 64x^2 - 36 = 0$, or $b = 36x^2 - 64 = 0$, or $a+b = 100x^2 - 100 = 0$. Solving each for x we find that the values that work are ± 1, $\pm\frac{3}{4}$, and $\pm\frac{4}{3}$.

S02S22 $9\bullet 2^{x+1} + 2^{1-x} = a$. Simplifying yields $18\bullet 2^x + \frac{2}{2^x} = a$ Let $y = 2^x$ then the equation becomes $18y + \frac{2}{y} = a$ or $18y^2 - a\bullet y + 2 = 0$. Use the quadratic formula to solve for y, $y = \frac{a \pm \sqrt{a^2 - 144}}{36}$ so $a = 12$.

S02S23

$$\sum_{n=1}^{2002} \sin\left(\frac{n\pi}{6}\right)\cos\left(\frac{n\pi}{6}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\frac{1}{2} + 0 + \frac{\sqrt{3}}{2}\left(\frac{-1}{2}\right) + \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right) + 0 + \ldots + \frac{-1}{2}\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{-\sqrt{3}}{2}\right)\frac{-1}{2} + 0 + \left(\frac{-\sqrt{3}}{2}\right)\frac{1}{2}$$

Every six terms the sum is zero. The last four terms remain $\therefore$ the sum is $\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}$

S02S24 The area of the triangle $\frac{1}{2}(base)(height)$ is the same no matter which altitude is used. If we call the respective sides $x, z,$ and w, then $A = \frac{5x}{2} = 2z = \frac{wy}{2}$ so $z = \frac{5x}{4}$. From the triangle inequality $x+z>w$, $x+w>z$ and $w+z>x$. $\therefore \frac{x}{4} < w < \frac{9x}{4}$. Since $\frac{5x}{2} = \frac{wy}{2}$, $\frac{20}{9} < y < 20 \therefore \frac{b}{a} = \frac{20}{\frac{20}{9}} = 9$

S02S25 $x^3-2x^2+20x-12=A(x-2)^3+B(x-2)^2+C(x-2)+D$. **Method I**: Use synthetic division on the polynomial with $x=2$ as the root repeatedly. The first remainder is D. The second remainder is C and so on. Yielding $A + C = 25$. **Method II**: Substitute $x=3$ into the equation, we get $A + B + C + D = 57$. Substitute $x=1$ into the equation, we get $-A + B + -C + D = 7$. Therefore $A + C = 25$.

S02S26 x must equal 24m +10 and 83n +10 to satisfy both equations. Which means the smallest value for x is 24(83) +10 =2002.

S02S27 $2-\sin A=\sqrt{2\cos^2 A+\sin A}=\sqrt{2-2\sin^2 A+\sin A}$. Let $x=\sin A$, $2-x=\sqrt{2-2x^2+x}$. Squaring both sides $x^2-4x+4=2-2x^2+x$ or $3x^2-5x+2=0 \therefore x=1,\frac{2}{3}$. Sin 2$A$=2sin$AcosA$=0, $\frac{\pm4\sqrt{5}}{9}$

S02S28 We use the equality $1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$.

$$\frac{1}{\sqrt{1^3}}+\frac{1}{\sqrt{1^3+2^3}}+...+\frac{1}{\sqrt{1^3+2^3+...+2002^3}}=\frac{2}{1(2)}+\frac{2}{2(3)}+...+\frac{2}{2002(2003)}=$$

$$=2\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+...+\frac{1}{2002}-\frac{1}{2003}\right)=2\left(\frac{2002}{2003}\right)=\mathbf{\frac{4004}{2003}}$$

S02S29 $1+2+2+3+3+3+...+n+n+...+n=1^2+2^2+3^2+...+n^2$

Method I: $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n \therefore a+b+c=1$.

Method II: Without a loss of generality, substitute $n=1$ and $a + b + c = 1$.

S02S30

$\log_{13}(\log_{11}(\log_7(\log_2 a)))=0 \therefore (\log_{11}(\log_7(\log_2 a)))=1 \therefore (\log_7(\log_2 a))=11$

$\therefore (\log_2 a)=7^{11} \therefore a=2^{7^{11}}$ Since 2 to a power has a four number cycle of 2,4, 8,6, then back to 2, we need only calculate the remainder of $7^{11}\div 4$. 7 mod 4 is 3 and 49 mod 4 is 1 then $7^{11} \bmod 4=3$. The units digit of a is the third number in the cycle, **8**.