New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest One** **Spring 2002**

S02J1. Compute the smallest positive integer with exactly 9 positive integral divisors.

S02J2. Compute all possible values for x: $\sqrt{\sqrt{x}-4}+\sqrt[4]{x}=2$

PART II: 10 minutes **NYCIML Contest One** **Spring 2002**

S02J3. a, b, and c are positive integers such that $a < b < c$ and $a! + b! + c! = 479001625$. Compute the value of a.

S02J4. The probability of having n broken eggs in a box of a dozen eggs is $\frac{n}{78}$.

Compute the probability of having exactly 17 broken eggs when 2 boxes of eggs are randomly chosen.

PART III: 10 minutes **NYCIML Contest One** **Spring 2002**

S02J5. For every tennis game that Andrei wins, he gets 120 points. For every game he loses, he loses n points. If n is an integer greater than one and Andrei starts with 0 points, compute the minimum value of n such that Andrei can get any integer score if he played enough games.

S02J6. Circles of radii 9, 9, 32, and x are mutually externally tangent. Compute all possible values of x.

ANSWERS:

S02J1. 36

S02J2. 16

S02J3. 1

S02J4. $\frac{134}{1521}$

S02J5. 7

S02J6. $\frac{32}{17}$

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2002**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02J7. A store sells Frisbees for \$7.00 each and Frosbees for \$10.00 each. The store sold at least one of each, and collected \$97.00. If x is the number of Frisbees sold, and y is the number of Frosbees sold, compute all possible ordered pairs (x, y).

S02J8. If the sum of the cubes of the roots of $x^2 - bx + 10 = 0$ is $6b$, compute the greatest possible value of b.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02J9.
$$a+b+c=2001,$$
$$a+b+d=2002,$$
$$a+c+d=2003,$$
$$b+c+d=2004.$$
Compute $d-c+b-a$.

S02J10. Compute all real values of x such that $|x+3|-|x-1|=x+1$.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2002**

S02J11. Nataliya works twice as fast as Boris does. If Boris worked for two more hours than it takes Nataliya to finish the job, he would do two thirds of the job. Compute how many hours it takes Boris to do the whole job.

S02J12. Given cube *ABCDEFGH* with side length 5, *I* is the midpoint of $\overline{AB}$. Let *J*, *K*, *L*, and *M* be the centers of squares *ADHE*, *DCGH*, *BCGF*, and *BFEA* respectively. Find the volume of pyramid *IJKLM*. (volume of a pyramid = $\frac{1}{3}hB$, h = height, B = area of the base)

ANSWERS:
S02J7. (1, 9), (11, 2)
S02J8. 6
S02J9. 2
S02J10. –5, -1, 3
S02J11. 12
S02J12. $\mathbf{\frac{125}{12}}$

New York City
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JUNIOR DIVISION **CONTEST NUMBER THREE** **SPRING 2002**
PART I: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02J13. There are 16 contestants in a women's tennis tournament. A player is eliminated only after she loses two matches. Compute the maximum number of matches that must be played to declare one winner. Assume there are no ties.
S02J14. Given isosceles trapezoid $ABCD$ with area 30, and P on base $\overline{AD}$, compute all values of x if $BA = CD$, $BC = x + 4$, $AP = 3$, $BP = x$, and $\overline{BP} \perp \overline{AD}$.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02J15. If $\underline{ABC}$ denotes the 3-digit number with non-zero digits A, B, and C (base 10), compute the largest prime number that can divide $\underline{ABC} + \underline{BCA} + \underline{CAB}$, for all possible $\underline{ABC}$.

S02J16. Given triangle ABC, angle bisector $AD = 10$, $AB = 16$, $AC = 9$. Find the length of $\overline{BC}$.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2002**

S02J17. Compute the remainder when 4^{2001} is divided by 15.

S02J18. Compute all real numbers x such that $\left\lfloor \frac{x+1}{4} \right\rfloor = \frac{3x-1}{4}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

ANSWERS:
S02J13. 31
S02J14. 3
S02J15. 37
S02J16. $\frac{25\sqrt{11}}{6}$
S02J17. 4
S02J18. $\frac{1}{3}$

S02J1. The number of positive integral divisors is found by taking the prime power decomposition of the integer, adding one to each exponent, and multiplying resulting results. We thus might try $2^8 = 256$ or $2^2 \cdot 3^2 = \mathbf{36}$.

S02J2. $\sqrt{\sqrt{x}-4}+\sqrt[4]{x}=2 \qquad \sqrt{\sqrt{x}-4}=2-\sqrt[4]{x} \qquad \left(\sqrt{\sqrt{x}-4}\right)^2=\left(2-\sqrt[4]{x}\right)^2$

$\sqrt{x}-4=4-4\sqrt[4]{x}+\sqrt{x} \qquad 4\sqrt[4]{x}=8 \qquad \sqrt[4]{x}=2 \therefore x=16$

S02J3. For all $x>1$, $x! = 1 \bullet 2 \bullet k$, therefore, even. Therefore either a, b, or c must be 1, since 0 is not a positive integer. Since all other positive integers are larger than 1, a must be **1**.

S02J4. To get 17 broken eggs, you can have 5 in box 1 and 12 in box 2. 6 in box 1 and 11 in box 2. … 12 in box 1 and 5 in box 2.

Thus, $P=2\left(\frac{5\cdot 12}{78^2}+\frac{6\cdot 11}{78^2}+\frac{7\cdot 10}{78^2}+\frac{8\cdot 9}{78^2}\right)=\frac{536}{6084}=\mathbf{\frac{134}{1521}}$

S02J5. If n and 120 have a greatest common factor d, Andrei's score will always be divisible by d, because if $n = dk$, $120 = dl$, and Andrei has W wins and L losses, his score is $120W - nL = dkW - dlL = d(kW - lL)$. If Andrei can get any integer score, d must be 1. The least $n > 1$ such that $(120, n) = 1$ is $n = \mathbf{7}$.

S02J6. $9^2 + (32+y)^2 = 41^2$

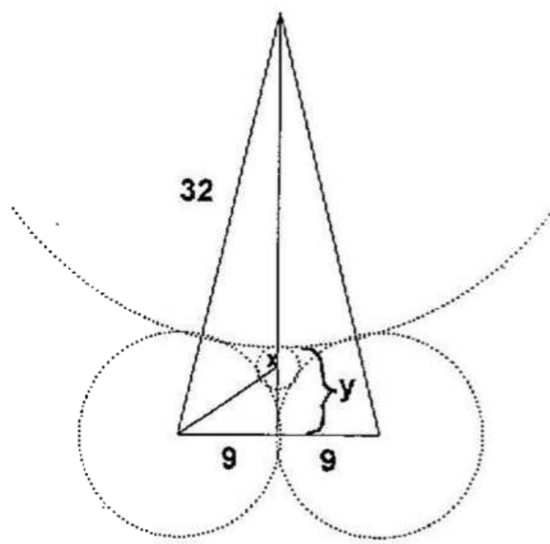
$32 + y = 40 \Rightarrow y = 8$

$(8-x)^2 + 9^2 = (9+x)^2$

$x^2 - 16x + 64 + 81 = x^2 + 18x + 81$

$64 = 34x$

$x = \mathbf{\frac{32}{17}}$



S02J7. $7x+10y=97$. $y>0$, $x>0$, so $y<9.7$. Only $y=2$ and $y=9$ give integral values of x, so the two ordered pairs are **(1, 9) and (11, 2)**.

S02J8. Let the 2 roots be m and n, then $mn = 10$, $m+n = b$, and $m^3 + n^3 = 6b$
$= (m+n)(m^2 - mn + n^2) = b(m^2 - mn + n^2)$
$\Rightarrow m^2 - mn + n^2 = 6$
$(m+n)^2 = m^2 + 2mn + n^2 = 6 + 3(10) = 36 = b^2$
$b = \pm 6 \Rightarrow$ the maximum of b is **6**.

S02J9. Subtracting the first equation from the second we get $d-c=1$. Subtracting the third equation from the fourth we get $b-a=1$. So $d-c+b-a=\mathbf{2}$.

S02J10. For $x\geq 1$, we get $x+3-(x-1)=x+1\Rightarrow x=3$
For $x<-3$, we get $-x-3-(-x+1)=x+1\Rightarrow x=-5$
For $x<1$ and $x\geq -3$, we get $x+3-(-x+1)=x+1\Rightarrow x=-1$
Thus the answers are $\{\mathbf{-5,-1,3}\}$.

S02J11. Let N be Nataliya's rate of work and B be Boris's rate of work. $N=2B$.
$B\left(\frac{1}{N}+2\right)=\frac{2}{3}$, or $B\left(\frac{1}{2B}+2\right)=\frac{2}{3}$, $B=\frac{1}{12}$. It will take Boris **12** hours to finish the job.

S02J12. Square $JKLM$ has area $\frac{25}{2}$, $\overline{IM}$ is perpendicular to Plane $JKLM$ and $IM=\frac{5}{2}$. So,
$\text{Volume}(IJKLM)=\frac{1}{3}(IM)(\text{Area}(JKLM))=\frac{1}{3}\left(\frac{5}{2}\right)\left(\frac{25}{2}\right)=\mathbf{\frac{125}{12}}$.

S02J13. Each person who loses the tournament lost exactly 2 matches. For the number of matches to be a maximum, the winner also had to lose one match. Therefore **31** matches were played.

S02J14. $\frac{(x+4+x+10)x}{2}=30$

$x^2+7x-30=0 \Rightarrow x=\mathbf{3}$

S02J15. $\underline{ABC}+\underline{BCA}+\underline{CAB} = 111(A+B+C) = 3\bullet 37\bullet(A+B+C)$
$A+B+C$ cannot exceed 37, so the largest prime factor is **37**.

S02J16. $\frac{AB}{AC}=\frac{BD}{DC}=\frac{16}{9}$. (Angle bisector theorem)
Let $DC = 9x$, and $BD = 16x$.
$AD^2 = AB\cdot AC - BD\cdot DC$ (Second angle bisector theorem, easily derived from Stewarts Theorem).
$100 = (16)(9) - (16x)(9x) = 144 - 144x^2$.
$144x^2 = 44$
$x^2 = \frac{44}{144} = \frac{11}{36}$
$x = \frac{\sqrt{11}}{6}$
$BC = DC + BD = 25x = \mathbf{\frac{25\sqrt{11}}{6}}$.

S02J17. $4^2 \equiv 1(\bmod 15)$, $4^{2000} \equiv 1^{1000}(\bmod 15) \equiv 1(\bmod 15)$,
$4^{2001} \equiv 4\cdot 4^{2000} \equiv 4\cdot 1(\bmod 15) \equiv 4(\bmod 15)$
The answer is **4**.

S02J18. Definition of $\lfloor x\rfloor$: $\lfloor x\rfloor$ is the integer that satisfies $0 \le (x - \lfloor x\rfloor) < 1$
$0 \le \frac{x+1}{4} - \frac{3x-1}{4} < 1$, $0 \le \frac{2-2x}{4} < 1$, $0 \le (1-x) < 2$, $-1 < x \le 1$
$0 < x+1 \le 2$, $0 < \frac{x+1}{4} \le \frac{1}{2}$
$0 = \left\lfloor \frac{x+1}{4} \right\rfloor = \frac{3x-1}{4}$
$x = \mathbf{\frac{1}{3}}$.