New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2001**

PART I: 10 minutes **NYCIML Contest One** **Fall 2001**

F01B1. Compute the smallest positive integer that is divisible by 11, but leaves a remainder of 1 when divided by 2, 3, 4, 5, or 6.

F01B2. Compute: $100^2-99^2+98^2-97^2+96^2-95^2+\ldots+2^2-1^2$.

PART II: 10 minutes **NYCIML Contest One** **Fall 2001**

F01B3. Compute x if $\frac{(x!)!}{x!}=119!$.

F01B4. Ming drives to work and averages 30 m.p.h. After the return trip along the exact same route, he calculates that he averaged 35 m.p.h. for the round trip. Compute his rate for the return trip.

PART III: 10 minutes **NYCIML Contest One** **Fall 2001**

F01B5. The sides of a triangle measure 5, 12, and 13. Compute the length of the radius of the inscribed circle.

F01B6. A rational number can be written as $.\bar{8}$ in base a and $.\bar{3}$ in base b, where a and b are positive integers. Express b in terms of a in simplest form.

ANSWERS

1. **121**
2. **5050**
3. **5**
4. **42**
5. **2**
6. $\frac{3a+5}{8}$

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SENIOR B DIVISION **CONTEST NUMBER TWO** **FALL 2001**

PART I: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01B7. Compute the area of a regular hexagon that is inscribed in a circle with radius 8.

F01B8. Compute all values of x:
$x^2+|3x|-10=0$.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01B9. Compute the sum of all positive integers less than 2001 which are multiples of 7.

F01B10. If $x+\frac{1}{x}=3$, compute $x^3+\frac{1}{x^3}$.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01B11. Compute the number of positive integral factors of 80^5.

F01B12. If $\log_{10} 2=a$ and $\log_{10} 3=b$, express $\log_5 72$ in terms of a and b with no logarithms.

ANSWERS

7. $96\sqrt{3}$
8. ± 2
9. 285285
10. 18
11. 126
12. $\frac{3a+2b}{1-a}$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2001**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2001**

F01B13. Three primes p, q and r, with $p < q < r$ have the property that $p + q = r$. Compute p.

F01B14. Compute the length of the diagonal of an isosceles trapezoid with sides 6,8,8 and 10.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2001**

F01B15. Three fair dice are thrown. If their sum is 6, compute the probability that they are all 2's.

F01B16. A painter can paint a room in 4 hours. An apprentice can paint the same room in 5 hours. Working together, compute the number of hours it would take 2 painters and 3 apprentices to paint the room?

PART III: 10 minutes **NYCIML Contest Three** **Fall 2001**

F01B17. Compute the base 5 number that is equivalent to 2001 base 10.

F01B18. ΔABC is an equilateral triangle with side 6. $\overline{BC}$ is extended through C to D so that $\overline{CD} = 6$. If E is the midpoint of $\overline{AB}$, and $\overline{DE}$ intersects $\overline{AC}$ at F, compute the area of quadrilateral $BEFC$.

ANSWERS

13. **2**

14. $2\sqrt{31}$

15. $\frac{1}{10}$

16. $\frac{10}{11}$

17. **31001**

18. $6\sqrt{3}$

SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2001**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2001**

F01B19. Each interior angle of a regular polygon measures 160°, compute the number of sides in the polygon.

F01B20. Compute the two prime factors of 9,991.

PART II: 10 minutes **NYCIML Contest Four** **Fall 2001**

F01B21. Compute the largest value of N such that 3^N is a factor of 30!

F01B22. Two different integers are chosen from the set of integers that are from 11 to 30 (including 11 and 30). Compute the probability that their product is odd.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2001**

F01B23. Compute the smallest positive integer N for which $90N$ is the cube of an integer.

F01B24. Compute the sum of the infinite series
$$\frac{2}{3}+\frac{4}{3^2}+\frac{6}{3^3}+\frac{8}{3^4}+\ldots+\frac{2N}{3^N}+\ldots .$$

<u>ANSWERS</u>

19. **18**
20. **97, 103**
21. **14**
22. $\mathbf{\frac{9}{38}}$
23. **300**
24. $\mathbf{\frac{3}{2}}$

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2001**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2001**

F01B25. Compute the sum of the coefficients of the terms of the expansion of $(1-2x)^7$.

F01B26. Find all possible ordered pairs of positive integers (x,y) which satisfy $x^2 - y^2 = 75$.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2001**

F01B27. How many different arrangements of the letters CLINTON starts with the two vowels? (IOCNTNL is one example).

F01B28. Compute the value of $(1 + \tan 10^\circ)(1 + \tan 35^\circ)$.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2001**

F01B29. A 10" x 10" x 10" cube is painted, then cut into 1000 1" x 1" x 1" cubes. How many of these smaller cubes are painted on exactly one face?

F01B30. The hands of a clock are perpendicular to each other exactly twice between 12 o'clock and 1 o'clock. To the nearest second, what is the difference in time between these two times?

ANSWERS

25. **-1**
26. **(38,37), (14,11), (10,5)**
27. **120**
28. **2**
29. **384**
30. **32 minutes 44 seconds**

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SENIOR B DIVISION **CONTEST NUMBER ONE** **Fall 2001**

SOLUTIONS

F01B1. **121**

The least common divisor of 2,3,4,5 and 6 is 60, but 61 is not divisible by 11. The next common divisor is 120, and 121 is divisible by 11.

F01B2. **5050**

Separating the pairs of terms and factoring,

$(100+99)(100-99) + (98+97)(98-97) + \ldots = 199 + 195 + .. + 7 + 3 = \frac{50}{2}(199+3) =$ 5050.

F01B3. **5** $\frac{(x!)!}{x!} = (x!-1)! = 119!$

$x! - 1 = 119$; $x! = 120$; $x=5$.

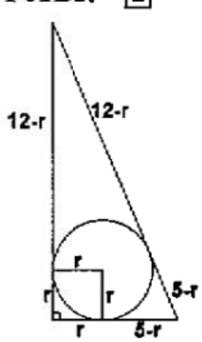
F01B4. **42**

Let D be the distance one way. Since

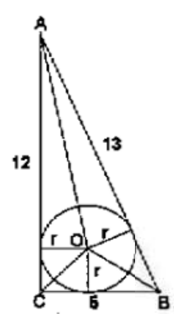
$\frac{D}{R} = T$, $\frac{2D}{35} = \frac{D}{30} + \frac{D}{X} \Rightarrow \frac{2}{35} = \frac{1}{30} + \frac{1}{X}$. Multiplying by $210X$, $12X = 7X+210$

$X = 42$.

F01B5. **2**



$5 - r + 12 - r = 13$
$17 - 2r = 13$
$r = 2$



$Area\Delta ABC = Area\,\Delta AOB$
$+ Area\Delta BOC$
$+ Area\Delta AOC$
$\frac{1}{2}(5r+12r+13r) = 30$
$15r = 30$
$r=2$

F01B6. $\boxed{\frac{3a+5}{8}}$

$$(.\bar{3})_b = \frac{3}{b} + \frac{3}{b^2} + \frac{3}{b^3} + \ldots = \frac{\frac{3}{b}}{1-\frac{1}{b}} = \frac{3}{b-1} \qquad (.\bar{8})_a = \frac{8}{a} + \frac{8}{a^2} + \frac{8}{a^3} + \ldots = \frac{\frac{8}{a}}{1-\frac{1}{a}} = \frac{8}{a-1}$$

$$\frac{8}{a-1} = \frac{3}{b-1} \qquad 8b-8 = 3a-3 \qquad b = \frac{3a+5}{8}$$

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SENIOR B DIVISION **CONTEST NUMBER TWO** **Fall 2001**

SOLUTIONS

F01B7. $\boxed{96\sqrt{3}}$

Since 6 equilateral triangles are formed when the radii are drawn, the area is $6\cdot\frac{8^2}{4}\sqrt{3}=96\sqrt{3}$.

F01B8. $\boxed{\pm 2}$

$(|x|+5)(|x|-2)=0$

$|x|=-5$ impossible

$|x|=2$ $x=\pm 2$.

F01B9. $\boxed{285{,}285}$

The smallest is 7, the largest is $7\cdot 285=1995$

$\frac{285}{2}(7+1995)=1001\cdot 285=285{,}285$.

F01B10. $\boxed{18}$

$x+\frac{1}{x}=3$. Squaring both sides, $x^2+2+\frac{1}{x^2}=9$, $x^2+\frac{1}{x^2}=7$. Multiply by $x+\frac{1}{x}$. $\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)=7\cdot 3$; $x^3+x+\frac{1}{x}+\frac{1}{x^3}=21$

$x^3+3+\frac{1}{x^3}=21$; $x^3+\frac{1}{x^3}=18$.

F01B11. $\boxed{126}$

$80^5=\left(2^4\cdot 5\right)^5=2^{20}\cdot 5^5$. The number of factors is the product of $(e_1+1)(e_2+1)\ldots$ where $e_1,e_2,\ldots$ are the exponents of the prime factors. $21\cdot 6=126$.

F01B12. $\boxed{\dfrac{3a+2b}{1-a}}$

Use the formula $\log_b a=\frac{\log_c a}{\log_c b}$

$$\log_5 72=\frac{\log_{10}72}{\log_{10}5}=\frac{\log_{10}9\cdot 8}{\log_{10}\frac{10}{2}}=\frac{2\log_{10}3+3\log_{10}2}{\log_{10}10-\log_{10}2}=\frac{3a+2b}{1-a}.$$

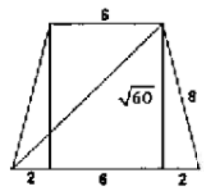
SOLUTIONS

F01B13. $\boxed{2}$ Since 2 is the only even prime, p must be 2.

F01B14. $\boxed{2\sqrt{31}}$

$\left(\sqrt{60}\right)^2 + 8^2 = d^2$

$d = \sqrt{124} = 2\sqrt{31}$.



F01B15. $\boxed{\frac{1}{10}}$ There are 3 possibilities, [1,2,3] or [2,2,2] or [1,1,4]. There are $\frac{3!}{2!} = 3$ permutations for [1,1, 4]; $3! = 6$ for [1,2,3]; and 1 for [2,2,2]. The probability is $\frac{1}{10}$.

F01B16. $\boxed{\frac{10}{11}}$ Let $\frac{x}{4}$ = part of the room one painter will paint, and $\frac{x}{5}$ = part of the room one apprentice will paint in x hours. $2\left(\frac{x}{4}\right) + 3\left(\frac{x}{5}\right) = 1$

$11x = 10$; $x = \frac{10}{11}$.

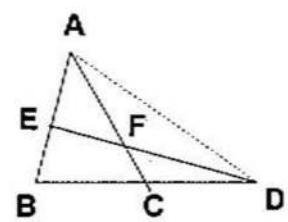
F01B17. $\boxed{31001}$

Divide 2001 by 5 and take the remainder in reverse order.

OR

Using the digits $5^0, 5^1, 5^2, 5^3$ and 5^4, 2001 is $3 \cdot 5^4 + 1 \cdot 5^3 + 1 \cdot 5^0$.

F01B18. $\boxed{6\sqrt{3}}$ Area of $\Delta ABC = \frac{6^2}{4}\sqrt{3} = 9\sqrt{3}$



Area of $\Delta ACD = 9\sqrt{3}$ since it has the same base and height.

$FC = \frac{1}{3}AC$ since medians intersect $\frac{2}{3}$ of the way to the opposite side.

Area $\Delta FCD = \frac{1}{3}\left(9\sqrt{3}\right) = 3\sqrt{3}$

Area $\Delta DBE = 9\sqrt{3}$ $\left(\frac{1}{2}\Delta ABD\right)$

Area $BEFC = 9\sqrt{3} - 3\sqrt{3} = 6\sqrt{3}$

Method II

The quadrilateral $BEFC$ can be split into ΔBEF and ΔBCF. Notice that F is the centroid of ΔABD, therefore ΔBEF and ΔBCF are each $\frac{1}{6}$ of ΔABD.

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Fall 2001**

SOLUTIONS

F01B19. **18**

$160 = \frac{180(N-2)}{N}$; $160N = 180N - 360$; $N = 18$.

OR

An exterior angle $= \frac{360}{N}$; $\frac{360}{N} = 20$; $N = 18$.

F01B20. **97,103**

$9991 = 100^2 - 3^2 = (100+3)(100-3) = (103)(97)$.

F01B21. **14**

3, 6, 12, 15, 21, 24 and 30 provide one 3
9, 18 provide 2
27 provide 3.
Total – 14 3's.

F01B22. $\boldsymbol{\frac{9}{38}}$

The product will only be odd if both integers are $\frac{{}_{10}C_2}{{}_{20}C_2} = \frac{45}{190} = \frac{9}{38}$.

F01B23. **300**

$90 = 2 \cdot 3^2 \cdot 5$. To be a perfect cube, each exponent must be a multiple of 3.
$N = 2^2 \cdot 3 \cdot 5^2 = 300$.

F01B24. $\boldsymbol{\frac{3}{2}}$

Set up the sum as an infinite number of infinite series.

$$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\ldots = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$$

$$+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\ldots = \frac{\frac{2}{9}}{1-\frac{1}{3}} = \frac{1}{3}$$

$$+\frac{2}{27}+\frac{2}{81}+\ldots = \frac{\frac{2}{27}}{1-\frac{1}{3}} = \frac{1}{9}$$

$+\ldots$

Then $1+\frac{1}{3}+\frac{1}{9}+\ldots = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **Fall 2001**

SOLUTIONS

F01B25. $\boxed{-1}$

The sum can be found by letting $x = 1$

$(1-2(1))^7 = -1$.

F01B26. $\boxed{(38,37),(14,11),(10,5)}$

$(x+y)(x-y) = 75$. There are 3 possibilities: $75\cdot 1$, $25\cdot 3$ and $15\cdot 5$. Each produces a solution: (38,37), (14, 11) and (10,5).

F01B27. $\boxed{120}$

Taking the consonants, there are $\frac{5!}{2!} = 60$ ways of arranging them. There are 2 ways of putting the two vowels in front of each arrangement. $60\cdot 2 = 120$.

F01B28. $\boxed{2}$

$\tan 45 = 1 = \tan(10+35) = \dfrac{\tan 10 + \tan 35}{1-\tan 10\cdot\tan 35}$

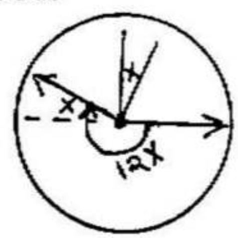
$1-\tan 10\cdot \tan 35 = \tan 10 + \tan 35$

$(1+\tan 10)(1+\tan 35) = 1 + \tan 10 + \tan 35 + \tan 10\cdot\tan 35 = 1 + 1 = 2$.

F01B29. $\boxed{384}$

The interior 8' x 8' square on each of the 6 faces will have one side painted $8\cdot 8\cdot 6 = 384$.

F01B30. $\boxed{\text{32 minutes 44 seconds}}$



In degrees, the minute hand moves 12 times as fast as the hour hand.

$12x - x = 180 \quad 11x = 180 \qquad x = \frac{180}{11}$

$12x = \frac{12(180)}{11}$ degrees the minute hand moves. Since 6° = 1 minute, there are $\frac{360}{11}$ minutes between the times. $\frac{360}{11} = 32\frac{8}{11}$ minutes = 32 minutes 44 seconds.