New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION | **CONTEST NUMBER ONE** | **FALL, 2001**

PART I | **FALL 2001** | **SENIOR A CONTEST 1** | **TIME: 10 MINUTES**

F01S1 Given $2a+b+c+d+e=10$, $a+2b+c+d+e=10$, $a+b+2c+d+e=20$, $a+b+c+2d+e=30$, $a+b+c+d+2e=50$. Compute $a\times b\times c\times d\times e$.

F01S2 Given the sequence $\sqrt[3]{10}, \sqrt[3]{100}, \sqrt[3]{1000}, \ldots, \sqrt[3]{10^n}, \ldots$ The product of the first k terms of the sequence is greater than one billion. Compute the smallest possible value for k.

PART II | **FALL 2001** | **SENIOR A CONTEST 1** | **TIME: 10 MINUTES**

F01S3 One tractor can plow a field in 15 hours. Another tractor plows the same field in 20 hours. Both tractors working together can plow the same field in h hours. Assuming the tractors plow at a constant rate, compute h.

F01S4 In ΔABC, $AB = 26$, $BC = 25$ and $CA = 17$. Compute the length of the triangle's longest altitude.

PART III | **FALL 2001** | **SENIOR A CONTEST 1** | **TIME: 10 MINUTES**

F01S5 A wooden cube has edges of length 5 inches. One-inch square holes, centered in each face, are drilled through to the opposite face. The edges of the holes are parallel to the edges of the cube. Compute the surface area of the resulting object in square inches.

F01S6 A unit circle is inscribed in equilateral triangle ABC and intersects side $\overline{AC}$ at D. A tangent to the circle is drawn parallel to side $\overline{BC}$ and intersects side $\overline{AC}$ at E. Compute $AE : ED$.

ANSWERS:

F01S1	0
F01S2	7
F01S3	$\frac{60}{7}$
F01S4	24
F01S5	192
F01S6	2:1

New York City Interscholastic Mathematics League

SENIOR A DIVISION	***CONTEST NUMBER TWO***	***FALL, 2001***

PART I	***FALL 2001***	***SENIOR A CONTEST 2***	***TIME: 10 MINUTES***

F01S7 A grandfather clock is set correctly to 12:00 at midnight on New Year's Eve 2001. The clock loses 3 minutes every 4 hours. Compute the time on the clock one year later.

F01S8 If $\sin\alpha + \cos\alpha = \sqrt{1.4}$, compute the value of $\sin^4\alpha + \cos^4\alpha$.

PART II	***FALL 2001***	***SENIOR A CONTEST 2***	***TIME: 10 MINUTES***

F01S9 Richard plants a Kalman tree. He measures the height after each year. The tree grows 7 inches during rainy years and 4 inches during dry years. After x years, the tree grew 84 inches. Compute the sum of all possible values of x.

F01S10 Two circles with centers O and P, each with a radius of 6, are tangent to each other at Q. The line $\overleftrightarrow{OP}$ intersects circle O at A and Q, and circle P at B and Q. Lines $\overleftrightarrow{ACD}$ and $\overleftrightarrow{BD}$ are tangent to the circle with center P at C and B respectively. Compute the length of $\overline{BD}$.

PART III	***FALL 2001***	***SENIOR A CONTEST 2***	***TIME: 10 MINUTES***

F01S11 Compute the length of the common chord of the two circles whose equations are $x^2 + y^2 - 2x - 5 = 0$ and $x^2 + y^2 - 2x + 8y - 13 = 0$.

F01S12 There are four consecutive positive integers, such that the smallest is a multiple of 3, the second is a multiple of 5, the third is a multiple of 7, and the largest is a multiple of 9. Compute the smallest possible sum of the four numbers.

ANSWERS:

F01S7	10:30 or 10:30 a.m.
F01S8	$\frac{23}{25}$
F01S9	66
F01S10	$6\sqrt{2}$
F01S11	$2\sqrt{5}$
F01S12	642

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER THREE** **FALL, 2001**

PART I ***FALL 2001*** ***SENIOR A CONTEST 3*** ***TIME: 10 MINUTES***

F01S13 Compute the sum of the coefficients of the polynomial expansion of $\left(x^2+2x+1\right)^4+\left(x^2-1\right)^4\left(5x^3-3x^2+8\right)$.

F01S14 Narrow's new pool is in the shape of two overlapping circles. A deck covers the overlap. If the circles are 8 feet apart and the radius of each circle is 8 feet, compute the surface area left for swimming.

PART II ***FALL 2001*** ***SENIOR A CONTEST 3*** ***TIME: 10 MINUTES***

F01S15 Each of the eight roots of $x^8 = 256$ may be written in the form $a+bi$, where a and b are real. Compute the product of all the roots for which $a \times b \neq 0$.

F01S16 Set S contains all possible three-digit numbers created by using the digits 1 to 9 without repetition. Compute the probability that a number chosen randomly from set S is divisible by 3.

PART III ***FALL 2001*** ***SENIOR A CONTEST 3*** ***TIME: 10 MINUTES***

F01S17 Compute $\dfrac{1}{3+\dfrac{1}{3+\dfrac{1}{3+\ldots}}}$.

F01S18 A function $f(x)$ satisfies: $f(x)+3f\left(\dfrac{1}{1-x}\right)=6x$ when $x \neq 1$.

Compute $f(3)$.

ANSWERS:

F01S13 256

F01S14 $\left(\dfrac{128\pi}{3}+64\sqrt{3}\right)$ *sq ft*

F01S15 16

F01S16 $\dfrac{5}{14}$

F01S17 $\dfrac{\sqrt{13}-3}{2}$

F01S18 2.25 or $\frac{9}{4}$ or $2\frac{1}{4}$

New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION **CONTEST NUMBER FOUR** **FALL, 2001**

PART I *FALL 2001* *SENIOR A CONTEST 4* *TIME: 10 MINUTES*

F01S19 Find the last two digits of 9^{2001}.

F01S20 A wooden cube has edges of length n inches ($n>1$). One-inch square holes, centered in each face, are drilled through to the opposite face. The edges of the holes are parallel to the edges of the cube. Express in simplest form the surface area of the resulting object, in square inches, in terms of n.

PART II *FALL 2001* *SENIOR A CONTEST 4* *TIME: 10 MINUTES*

F01S21 David and Toni are 81 miles apart and start walking toward each other at the same time. David walks at a constant rate of 3 miles per hour. Toni walks 1 mile the first hour, 1 ½ miles the second hour, and 2 miles the third hour, and so on. Compute the number of hours it will take for Toni and David to meet.

F01S22 Points A (7, 13) and B (9, 17) are on a circle with center (h, k) and radius 5. Compute all possible coordinates (h, k).

PART III *FALL 2001* *SENIOR A CONTEST 4* *TIME: 10 MINUTES*

F01S23 If 6 geese can lay 30 golden eggs in 7 days, compute the time, in days, needed for 8 geese to lay 18 golden eggs.

F01S24 Given that $xyz \neq 0$ and $\log\ (xyz) = a$, $\log\left(\frac{xy}{z}\right) = b$, $\log\left(\frac{xz}{y}\right) = c$. Express $\frac{yz}{x}$ in simplest form, in terms of a, b, and c, with no logarithms.

ANSWERS:

F01S19	09 both digits required
F01S20	$6n^2 + 12n - 18$
F01S21	12
F01S22	(4, 17) and (12, 13) both required
F01S23	63 / 20 or 3.15
F01S24	10^{a-b-c}

New York City Interscholastic Mathematics League

SENIOR A DIVISION — CONTEST NUMBER FIVE — FALL, 2001

PART I — ***FALL 2001*** — ***SENIOR A CONTEST 5*** — ***TIME: 10 MINUTES***

F01S25 Sal is dealt 5 cards from a standard 52-card deck. They are the three and four of hearts, five and six of spades and the nine of clubs. If he discards the nine and receives a different card from the deck, compute the probability that Sal will complete the straight (5 cards in numerical order).

F01S26 The graphs of $y=|x|-4$ and $y=-|x+2|$ partition the plane into several regions. Compute the area of the bounded region.

PART II — ***FALL 2001*** — ***SENIOR A CONTEST 5*** — ***TIME: 10 MINUTES***

F01S27 $\left(1-\frac{1}{2}\right)\left(2-\frac{2}{3}\right)\left(3-\frac{3}{4}\right)\cdots\left(2001-\frac{2001}{2002}\right)=\frac{x!}{2002}$.
Compute x.

F01S28 If $\log_a\left(b^3\right)+\log_b\left(a^3\right)=10$, then b may be expressed in terms of a in two distinct ways. Determine these two expressions of b in simplest form.

PART III — ***FALL 2001*** — ***SENIOR A CONTEST 5*** — ***TIME: 10 MINUTES***

F01S29 A regular n-sided polygon is inscribed in a circle. The measure of each of the angles of the polygon is an integral number of degrees. If two vertices of the n-gon intercept an arc of $60°$, compute the number of distinct values of n.

F01S30 If x is a real number and $x \neq -3$, compute the minimum value of the expression
$$\frac{x^2+2x-1}{(x+3)^2}.$$

ANSWERS:

F01S25	$\frac{8}{47}$
F01S26	6
F01S27	2001
F01S28	$b=a^3$ and $b=a^{\frac{1}{3}}$
F01S29	12
F01S30	-1

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

Senior A Division **Solutions** **CONTEST NUMBER 1** **FALL, 2001**

F01S1 **Answer:** 0. By adding the five equations, we get $6(a+b+c+d+e) = 120$. Divide by 6, so $a+b+c+d+e = 20$. $a = -10$, $b = -10$, $c = 0$, $d = 10$, and $e = 30$. $a \times b \times c \times d \times e = 0$.

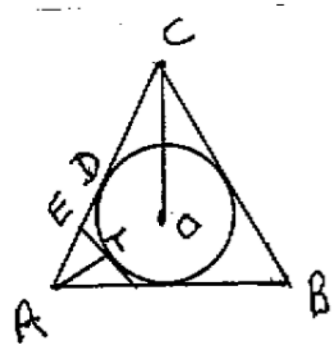
F01S2 **Answer:** 7. The product of the first k terms of the sequence may be written as $10^{\frac{1}{3}} \times 10^{\frac{2}{3}} \times 10^{\frac{3}{3}} \times \ldots \times 10^{\frac{k}{3}}$, so if $10^{\frac{1}{3}} \times 10^{\frac{2}{3}} \times 10^{\frac{3}{3}} \times \ldots \times 10^{\frac{k}{3}} > 1{,}000{,}000{,}000 = 10^9$ then $\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \ldots + \frac{k}{3} > 9$. $\therefore \frac{k(k+1)}{2 \times 3} > 9$ or $k(k+1) > 54$. The smallest possible value for k is 7.

F01S3 **Answer:** $\frac{60}{7}$. $\frac{h}{15} + \frac{h}{20} = 1$. $\frac{35h}{300} = 1$ $\therefore h = \frac{60}{7}$ hours.

F01S4 **Answer:** 24. Use Heron's formula for the area of a triangle, $K = \sqrt{s(s-a)(s-b)(s-c)}$ where s = the semiperimeter, and a, b, c are the lengths of the three sides. Since $s = 34$, K = 204. The area may also be computed using $K = \frac{bh}{2} = \frac{17h}{2} = 204$ therefore $h = 24$.

F01S5 **Answer:** 192. There are two parts to the surface area, the face and the hole. On each face we have 24 in^2. The face has an original area of 25 in^2 and we subtract 1 in^2 for the hole. For each hole drilled through a face to the center, we gain an additional 8 in^2. Therefore there are 32 in^2 for each side for a total surface area of $6 \cdot 32 = 192$ in^2.

F01S6 **Answer:** 2:1. Angles A, B, and C are 60 degrees. The line segment from the center of the inscribed circle to the vertices bisects these angles so angles OCD and TAE are 30 degrees. Triangle EAT is a 30-60-90 triangle. $TE = DE$, since they are tangent segments from an external point to the same circle. $AE : TE = AE : DE = 2 : 1$.



NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

Senior A Division **Solutions** Contest Number 2 **FALL, 2001**

F01S7 **Answer:** 10:30. $\frac{3\text{ min}}{4\text{hours}} = \frac{18\text{ min}}{24\text{hours}} = \frac{720\text{ min or }12\text{ hours}}{40\text{days}}$. So every 40 days the clock is correct at 12:00. After 360 days, the clock shows the correct time of 12:00. In the last five days the clock loses 90 minutes total. Instead of showing 12:00, the clock will read 10:30.

F01S8 **Answer:** .92 or 23 / 25. $(\sin\alpha + \cos\alpha)^2 = \sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha$

$1.4 = 1 + 2\sin\alpha\cos\alpha$ and therefore $\sin\alpha\cos\alpha = .2$

$(\sin^2\alpha + \cos^2\alpha)^2 = \sin^4\alpha + \cos^4\alpha + 2\sin^2\alpha\cos^2\alpha$

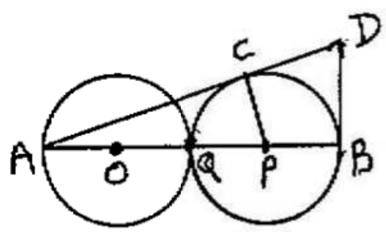
$1 = \sin^4\alpha + \cos^4\alpha + 2(.2)^2$

$\sin^4\alpha + \cos^4\alpha = 1 - .08 = .92$

F01S9 **Answer:** 66. Let a and b equal the number of rainy and dry years respectively. Then $x = a + b$. Since $7a + 4b = 84$, where a and b are non-negative integers, the only pairs (a, b) that work are (12, 0), (8, 7), (4, 14), and (0, 21), so $x = a + b = 12, 15, 18$ or 21. The sum of all possible values of x is 66.

F01S10 **Answer:** $6\sqrt{2}$. AB = 24, CP = 6, AP = 18. Use the Pythagorean theorem on triangle PAC yields CA=$12\sqrt{2}$. $\frac{BD}{AB} = \frac{PC}{CA}$ because triangle ABD is similar to triangle ACP.

$\therefore \frac{x}{24} = \frac{6}{12\sqrt{2}}$ so $x = 6\sqrt{2}$



F01S11 **Answer:** $2\sqrt{5}$. Subtract the two equations from each other to get $8y - 8 = 0$. So $y = 1$; by substitution $x^2 - 2x - 4 = 0$. Thus $x = 1 \pm \sqrt{5}$. The distance between $(1+\sqrt{5}, 1)$ and $(1-\sqrt{5}, 1)$ is $2\sqrt{5}$.

F01S12 **Answer:** 642. Let the four consecutive numbers be a, $a + 1$, $a + 2$, $a + 3$. Then the numbers $2a$, $2a + 2$, $2a + 4$, $2a + 6$ are divisible by 3, 5, 7, and 9 respectively, so $2a - 3$ is divisible by 3, 5, 7, and 9. The smallest possible positive value of a for which this is true occurs when $2a - 3 = \text{l.c.m.}(3,5,7,9) = 315$, or when $a = 159$. Then the sum of the four numbers is $159 + 160 + 161 + 162 = 642$.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

Senior A Division Solutions CONTEST NUMBER 3 FALL, 2001

F01S13 **Answer:** 256. By substituting $x = 1$, we obtain the sum of the coefficients of the expanded polynomial: $4^4 = 256$.

F01S14 **Answer:** $\frac{128\pi}{3} + 64\sqrt{3}$. The area available for swimming is equal to twice the area of the circles minus twice the overlapping area. The overlapping region is twice the difference of the area of a 120° sector of the circle and the triangle formed by the radii and the common chord. The area of the sector is one-third the area of the circle. The triangle has an area of ½(8)(8)sin 120°. Therefore the area is $\frac{128\pi}{3} + 64\sqrt{3}$.

F01S15 **Answer:** 16. The product of all the roots is -256. The roots that have a = 0 or b = 0, are 2, -2, $2i$, and $-2i$. Their product is -16. Therefore the remaining roots have a product of 16.

F01S16 **Answer:** $\frac{5}{14}$. To be divisible by 3, the sum of the digits must be divisible by 3 (0 in mod 3). The digits one to nine in mod 3 are equal to 1, 2, 0, 1, 2, 0, 1, 2, 0. The only combinations that work are (1, 1, 1), (0, 0, 0), (2, 2, 2) and (1, 2, 0). As no digits can repeat, each of the first three cases can be obtained in $3 \times 2 \times 1 = 6$ ways. The last case can be done in $3 \times 3 \times 3 \times {}_3P_3 = 162$ ways, so there are 162 + 18 = 180 numbers in set S that are divisible by 3. Since there are $9 \times 8 \times 7 = 504$ numbers total in set S, the probability that one chosen at random is divisible by 3 is $\frac{180}{504} = \frac{5}{14}$.

F01S17 **Answer:** $x = \frac{\sqrt{13} - 3}{2}$. Let $x = \cfrac{1}{3 + \cfrac{1}{3 + \dots}}$ then $x = \frac{1}{3 + x}$.

Cross-multiply and solve for x:

$x^2 + 3x - 1 = 0 \quad \therefore x = \frac{\sqrt{13} - 3}{2}$. (The other root to the quadratic is negative but $x>0$).

F01S18 **Answer:** 2.25. Substitute $x = 3$. $f(3) + 3f\left(\frac{-1}{2}\right) = 18$

Next substitute $x = -½$ to get $f\left(\frac{-1}{2}\right) + 3f\left(\frac{2}{3}\right) = -3$

Then substitute $x = 2/3$, and get $f\left(\frac{2}{3}\right) + 3f(3) = 4$.

Add the three equations together to get $4\left(f(3) + f\left(\frac{-1}{2}\right) + f\left(\frac{2}{3}\right)\right) = 19$. Substituting for

$f\left(\frac{2}{3}\right)$ and $f\left(\frac{-1}{2}\right)$, yields $f(3) + \left(\frac{18 - f(3)}{3}\right) + (4 - 3f(3)) = \frac{19}{4}$. $\therefore f(3) = 2.25$.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

Senior A Division　　Solutions　　Contest Number 4　　FALL, 2001

F01S19　**Answer:** 09. In order to find the last two digits of 9^{2001} we write it as $(10-1)^{2001}$

$$=10^{2001}+\binom{2001}{1}10^{2000}(-1)^1+\binom{2001}{2}10^{1999}(-1)^2+\ldots+\binom{2001}{1999}10^2(-1)^{1999}+\binom{2001}{2000}10(-1)^{2000}+(-1)^{2001}$$

Every term in the above sum is divisible by 100 except for the last two terms, which add to $20010-1=20009$. Therefore the last two digits of 9^{2001} are 09.

F01S20　**Answer:** $6n^2+12n-18$. There are six square faces. Each face has a surface area of n^2-1. There are three holes of length n. Each hole is rectangular with the middle missing. The interior surface area created by the hole is $3\cdot 4$ sides $\cdot(n-1)$.
The total surface area is $6n^2+12n-18$.

F01S21　**Answer:** 12. The distance traveled by both walkers is $4+4.5+5+\ldots+(4+½(n-1))$. The total distance is $\frac{n}{2}\left(8+\frac{1}{2}(n-1)\right)=81$. Therefore $n^2+15n-324=(n+27)(n-12)=0$.
Thus $n=12$ hours.

F01S22　**Answer:** (4, 17), (12, 13) (both needed). Let the center of the circle be (h, k). By using the standard form of the equation of a circle, $(x-h)^2+(y-k)^2=r^2$, we obtain $(7-h)^2+(13-k)^2=25=(9-h)^2+(17-k)^2$. This implies that $h+2k=38$. Substituting $h=38-2k$ and solving for k gives us $k=13$ or 17. When $k=13$, $h=12$ and when $k=17$, $h=4$.

F01S23　**Answer:** $\frac{63}{20}$. It takes $6\text{ geese}\times 7\text{ days}$ to produce 30 eggs, or $\frac{5}{7}$ eggs per goose $\times$ day. Since there are 8 geese and x days, they produce $\frac{40x}{7}=18$ eggs.

Therefore $x=\frac{63}{20}$ days.

F01S24　**Answer:** 10^{a-b-c}. $\log(xyz)=a,\quad \log\left(\frac{xy}{z}\right)=b,\quad \log\left(\frac{xz}{y}\right)=c$.

$$\log(xyz)=\log x+\log y+\log z=a$$

$$\log\left(\frac{xy}{z}\right)=\log x+\log y-\log z=b$$

$$\log\left(\frac{xz}{y}\right)=\log x+\log z-\log y=c$$

By adding the last two equations, we get $2\log(x)=b+c$.
Subtracting this equation from the first equation we get

$$\log y+\log z-\log x=\log\left(\frac{yz}{x}\right)=a-b-c\quad\therefore\frac{yz}{x}=10^{a-b-c}.$$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

Senior A Division Solutions CONTEST NUMBER 5 FALL, 2001

F01S25 **Answer:** $\frac{8}{47}$. Sal must receive either a two or a seven to get a straight. There are 4 twos and 4 sevens remaining in the deck. Since Sal used the five cards in his hand, there are 47 cards left from which to choose. The probability of getting the straight is $\frac{8}{47}$.

F01S26 **Answer:** 6. The equations $y=|x|-4$ and $y=-|x+2|$ form the boundary of the region.
$y=|x|-4$ may be expressed as $y=x-4$ for $x\geq 0$ and $y=-x-4$ for $x<0$.
$y=-|x+2|$ may be expressed as $y=x+2$ for $x\leq -2$ and $y=-x-2$ for $x>-2$.
The slopes of the equations are negative reciprocals so the shape is rectangular.
The intersections of $y=|x|-4$ and $y=-|x+2|$ are (-3,-1) and (1, -3).
The other vertices of the rectangle are the corners of the graphs of the absolute value equations (0, -4) and (-2, 0). Therefore the area of the rectangle is 6.

F01S27 **Answer:** 2001. We have $n-\frac{n}{n+1}=\frac{n(n+1)-n}{n+1}=\frac{n^2}{n+1}$, so the series may be transformed into $\frac{1^2}{2}\times\frac{2^2}{3}\times\frac{3^2}{4}\times\frac{4^2}{5}\times\ldots\times\frac{2001^2}{2002}=\frac{(2001!)^2}{(2001!)2002}=\frac{2001!}{2002}$, so $x=2001$.

F01S28 **Answer:** $b=a^3$ or $b=a^{1/3}$. If we let $x=\log_a b$, then $\log_a(b^3)=3\log_a b=3x$ and $\log_b(a^3)=3\log_b a=\frac{3}{x}$. $\therefore 3x+\frac{3}{x}=10$ or $x+\frac{1}{x}=\frac{10}{3}$. Multiply both sides by $3x$, $3x^2-10x+3=0$ so $x=1/3$ or 3. Now substitute for x above and solve for b.

F01S29 **Answer:** 12. For a regular n-gon to have angles of integral measures, n must divide 360°. The measure of an interior angle of a regular n-gon is $180-\frac{360}{n}$ so n must divide 360. To intercept an arc of 60°, $\frac{360}{n}$ must divide 60. So $\frac{n}{6}$ must be an integer. Since n divides 360 and 6 divides n, n may be any of the factors of 60 multiplied by 6. n may be 6, 12, 18, 24, 30, 36, 60, 72, 90, 120, 180, or 360.

F01S30 **Answer:** -1. Let $a=\frac{x^2+2x-1}{(x+3)^2}$. Cross-multiply to obtain $a(x+3)^2=x^2+2x-1$.
Writing as a quadratic in x: $(a-1)x^2+(6a-2)x+(9a+1)=0$. Because x is real, the discriminant $D=(6a-2)^2-4*(a-1)*(9a+1)=8a+8\geq 0$. Therefore $a\geq -1$.