New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2001**

PART I: 10 minutes **NYCIML Contest One** **Fall 2001**

F01J1. Danny is thinking of the smallest positive integer greater than one that leaves a remainder of 1 when divided by either 3,4,6, or 8. Compute the number.

F01J2. Compute the ordered pair (a, b) if a and b are the nonzero roots of $x^2+\frac{a}{2}x+2b=0$.

PART II: 10 minutes **NYCIML Contest One** **Fall 2001**

F01J3. Compute the probability that the sum of the digits of a 3-digit integer is even.

F01J4. Points A (46, 34) and D (93, 21) are opposite vertices of regular hexagon $ABCDEF$. Compute (x, y) where x is the sum of the x-coordinates of the other four vertices and y is the sum of the y-coordinates of those vertices.

PART III: 10 minutes **NYCIML Contest One** **Fall 2001**

F01J5. Jan can paint a room in 11 hours, working alone. Peter can paint the same room in 9 hours, working alone. Jan starts painting the room and after 3 hours Peter joins him. Compute the number of hours after Peter joins that the room will be completely painted.

F01J6. In quadrilateral $ABCD$, $AB = BC = \sqrt{2}$, $CD = 1$, and $DA = \sqrt{5}$. If angle ABC is a right angle, compute the area of quadrilateral $ABCD$.

ANSWERS:
F01J1. **25**
F01J2. **(2, -3)**
F01J3. $\mathbf{\frac{1}{2}}$
F01J4. **(278, 110)**
F01J5. $\mathbf{\frac{18}{5}}$
F01J6. **2**

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2001**

PART I: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01J7. Compute the maximum possible value of the ratio of a three-digit number to the sum of its digits.

F01J8. $\overline{AB}$ and $\overline{CD}$ are the bases of trapezoid $ABCD$. If $AB = 18$, $BC = 17$, and $DA = 10$, compute the minimum length of $\overline{CD}$, if the height of the trapezoid is 8.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01J9. In right triangle ABC, angle C is a right angle. $AC = 9$ and altitude $CD = 3$. Compute BC.

F01J10. Compute all integer solutions (x, y) for the equation $y \cdot |x| + 7 \cdot |x| - 2y - 15 = 0$.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2001**

F01J11. If a regular hexagon with a side of length 1 is inscribed in a circle, compute the area outside the hexagon that is within the circle.

F01J12. If a and b are roots of the equation $x^2 - 3x + 1 = 0$ and $a > b$, compute $a^8 - b^8$.

ANSWERS:

F01J7. **100**

F01J8. **9**

F01J9. $\mathbf{\frac{9\sqrt{2}}{4}}$

F01J10. **(1, -8), (-1, -8), (3, -6), (-3, -6).**

F01J11. $\pi - \frac{3\sqrt{3}}{2}$

F01J12. $987\sqrt{5}$

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JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 2001**

PART I: 10 minutes | **NYCIML Contest Three** | **Fall 2001**

F01J13. Compute the minimum possible value of the ratio of a three-digit number to the sum of its digits.

F01J14. If a, b, c, and d are all positive, $a + b = c + d$, and $ab > cd$,

compute: $\dfrac{(a^2-c^2)+(b^2-d^2)}{ab-cd}$.

PART II: 10 minutes | **NYCIML Contest Three** | **Fall 2001**

F01J15. Compute the number of positive integers smaller than or equal to 100 that have an odd number of positive factors.

F01J16. There are 12 adjacent parking spaces in a parking lot and 8 of them are occupied. A large truck arrives, needing 2 adjacent unoccupied spaces to park. Compute the probability that it will be able to park. (Each arrangement of cars is equally likely.)

PART III: 10 minutes | **NYCIML Contest Three** | **Fall 2001**

F01J17. Compute the remainder when $1+6+6^2+6^3+...+6^{2002}+6^{2003}$ is divided by 43.

F01J18. Given a 4 x 4 tic – tac – toe board, how many different ways can you arrange 'X's and 'O's so that in each row and in each column there are exactly 2 'X's and 2 'O's?

ANSWERS:

F01J13. $\mathbf{\dfrac{199}{19}}$

F01J14. **–2**

F01J15. **10**

F01J16. $\mathbf{\dfrac{41}{55}}$

F01J17. **0**

F01J18. **90**

F01J1. 25. The smallest positive integer that is divisible by either 3, 4, 6, or 8 is 24. Danny is thinking of 24 + 1 = 25.

F01J2. (2, -3). $a+b=-\frac{a}{2}$, $ab=2b \rightarrow a=2$, $2+b=-\frac{2}{2} \rightarrow 2+b=-1$, $b=-3$

Thus, $a=2$, $b=-3$

F01J3. $\frac{1}{2}$. There are 9 possibilities for the 100's digit, and ten possibilities for the tens and units digit, for 900 possible 3-digit numbers. The sum of the digits will be even if we have 3 even digits or 1 even digit $= 4\cdot5\cdot5+4\cdot5\cdot5+5\cdot5\cdot5+5\cdot5\cdot5=450$. $\frac{450}{900}=\frac{1}{2}$.

F01J4. (278, 110). The midpoint of $\overline{AD}$ is $\left(\frac{93+46}{2},\frac{21+34}{2}\right)=\left(\frac{139}{2},\frac{55}{2}\right)$, which is also the midpoint of $\overline{BE}$ and $\overline{CF}$, thus $(x,y)=\left((4)\frac{139}{2},(4)\frac{55}{2}\right)=(278,110)$.

F01J5. $\frac{18}{5}$. Jan paints $\frac{1}{11}$ of the room in an hour, Peter paints $\frac{1}{9}$ of the room in an hour. After Jan works for 3 hours, $\frac{8}{11}$ of the room remains to be painted. It will take Jan and Peter

$\frac{\frac{8}{11}}{\frac{1}{11}+\frac{1}{9}}=\frac{18}{5}$ hours to finish.

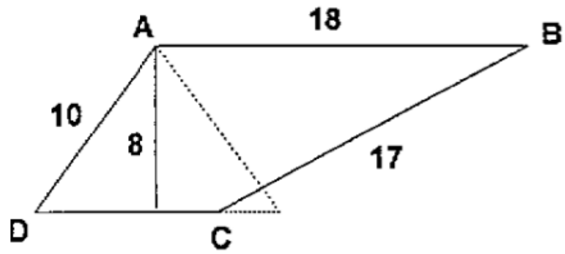
F01J6. 2. Split the quadrilateral into two triangles by diagonal $\overline{AC}$. The area of ΔABC is 1. Using the Pythagorean Theorem, we see that $AC=2$. Observe that $AC^2+CD^2=DA^2$. Therefore, $\angle ACD$ is a right angle. The area of ΔACD is therefore also 1. Adding, the area of quadrilateral $ABCD$ is 2.

F01J7. 100. Let $\underline{abc}$ be a three digit number. The ratio is

$$\frac{100a+10b+c}{a+b+c} \le \frac{100a+100b+100c}{a+b+c} = 100.$$

Equality occurs when $b = c = 0$.

F01J8. 9. To minimize CD, we want as many of the legs to slant towards the center of $\overline{AB}$ as possible. However, that would make the legs intersect. Therefore, we want to slant $\overline{BC}$ inside and $\overline{DA}$ outside, making



$$CD = 18 - \sqrt{17^2 - 8^2} + \sqrt{10^2 - 8^2}$$
$$= 18 - 15 + 6 = 9.$$

F01J9. $\frac{9\sqrt{2}}{4}$. $AD = \sqrt{81-9} = 6\sqrt{2}$

Since ΔADC is similar to ΔACB, $\frac{BC}{AC} = \frac{CD}{AD}$. $BC = \frac{9(3)}{6\sqrt{2}} = \frac{9\sqrt{2}}{4}$.

F01J10. (1, -8), (-1, -8), (3, -6), (-3, -6). If $x \ge 0$, we have $xy + 7x = 2y + 15$.

$$x = \frac{2y+15}{y+7} = 2 + \frac{1}{y+7}$$

Since x is an integer, $y = -6$ or -8, and the corresponding x values are 3 and 1 respectively.

If $x < 0$, we have $-xy - 7x = 2y + 15$. $x = -\frac{2y+15}{y+7} = -2 - \frac{1}{y+7}$

Again, since x is an integer, $y = -6$ or -8, and the corresponding x values are -3 and -1 respectively.

Thus, the 4 solutions are (1, -8), (-1, -8), (3, -6), (-3, -6).

F01J11. $\pi - \frac{3\sqrt{3}}{2}$. Area(hexagon) = 6 Area(triangle) = $\frac{\sqrt{3}}{4}(6)(1) = \frac{3\sqrt{3}}{2}$.

Area(circle) = $\pi r^2 = \pi$

Thus, the answer is $\pi - \frac{3\sqrt{3}}{2}$.

F01J12. $987\sqrt{5}$. Sum of roots: $a+b = 3$. Product of roots: $ab = 1$

$(a+b)^2 = 9 = a^2 + 2ab + b^2$

$7 = a^2 + 2ab + b^2 - 2ab = a^2 + b^2$

$(a^2+b^2)^2 = 49 = a^4 + 2a^2b^2 + b^4$

$47 = a^4 + 2a^2b^2 + b^4 - 2a^2b^2 = a^4 + b^4$

$(a^4+b^4)^2 = 2209 = a^8 + 2a^4b^4 + b^8$

$2205 = a^8 + 2a^4b^4 + b^8 - 4a^4b^4 = a^8 - 2a^4b^4 + b^8 = (a^4 - b^4)^2$

$a^4 - b^4 = \sqrt{2205} = 21\sqrt{5}$

$a^8 - b^8 = (a^4 + b^4)(a^4 - b^4) = 987\sqrt{5}$.

F01J13. $\frac{\mathbf{199}}{\mathbf{19}}$**.** The ratio will be a minimum when the hundred's digit is as small as possible and the tens and units digit are as large as possible. This occurs when the number is 199 and the ratio is $\frac{199}{1+9+9}=\frac{199}{19}$

F01J14. −2. $\frac{(a^2-c^2)+(b^2-d^2)}{ab-cd}=\frac{(a+b)^2-2ab-(c+d)^2+2cd}{ab-cd}=\frac{-2(ab-cd)}{(ab-cd)}=-2$.

F01J15. 10 If a number has an odd # of positive factors, the number is a perfect square. $100=10^2$. Thus, the answer is 10.

F01J16. $\frac{\mathbf{41}}{\mathbf{55}}$**.** We will compute the probability that the truck cannot park. This will occur if the 4 spots are not adjacent. This is equivalent to counting the number of strings with 8 x's and 4 o's with the o's not adjacent. This is equivalent to counting the number of strings with 9 x's and 4 o's with the o's not adjacent and an x last. This is equivalent to counting the number of strings with 5 x's and 4 o's There are $\binom{9}{4}$ such strings. The probability the truck cannot park is

$\frac{\binom{9}{4}}{\binom{12}{4}}=\frac{9!8!4!}{5!4!12!}=\frac{14}{55}$. The probability the truck can park is $1-\frac{14}{55}=\frac{41}{55}$

F01J17. 0. Notice that $1+6+6^2=43$, every 3 consecutive terms add up to a multiple of 43.
$S=(1+6+6^2)(1+6^3+6^6+6^9+\ldots+6^{1998}+6^{2001})\equiv 0 \pmod{43}$.

F01J18. 90. There are ${}_4C_2=6$ different rows. If the first two rows are the same, the bottom 2 rows are uniquely determined (this can be done in 6 ways). If there is exactly 1 column with 2 'X's in the first two rows, there are 2 possibilities for the bottom 2 rows (this can be done in 6 × 2 × 2 × 2 ways.) If there are no columns with 2 'X's, the bottom two rows must also be such that they have totally different columns (this can be done in 6 x 6 ways). 6 + 48 +36 = 90.