New York City Interscholastic Mathematics League

SENIOR B DIVISION | **CONTEST NUMBER ONE** | **SPRING 2001**

PART I: 10 minutes | **NYCIML Contest ONE**

S01B1. Compute the units digit of 7^{2001}.

S01B2. Solve for all values of x: $x^{\frac{1}{3}} + 6 = -\frac{8}{x^{\frac{1}{3}}}$.

PART II: 10 minutes | **NYCIML Contest One** | **Spring 2001**

S01B3. Compute the sum of the coefficients of the terms in the expansion of $(x+y)^8$.

S01B4. Express in simplest form $\frac{2001! - 2000!}{2002! - 2(2000!)}$.

PART III: 10 minutes | **NYCIML Contest One** | **Spring 2001**

S01B5. Compute the sum of the infinite series $.1 + .01 + .001 + .0001 + \ldots$.

S01B6. A point P is chosen in the interior of rectangle $ABCD$ so that $PA = 4$, $PB = 5$, and $PC = 6$. Compute PD.

ANSWERS

1. 7
2. $-8, -64$
3. 256
4. $\frac{1}{2003}$
5. $\frac{1}{9}$
6. $3\sqrt{3}$

New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER TWO** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest TWO**

S01B7. Compute $\sqrt{5^5+5^5+5^5+5^5+5^5}$.

S01B8. In a circle, a 6-inch chord and a 14-inch chord are parallel and 10 inches apart. Compute the number of inches in the radius of the circle.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2001**

S01B9. Compute how many positive integers less than 1000 are divisible by 3, but not divisible by 5.

S01B10. The first term of a geometric progression is 1 and the fourth term is 7. Compute the product of the first four terms.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2001**

S01B11. Three times the third term of an arithmetic progression is equal to 6 times the sixth term. Compute the ninth term.

S01B12. The measure of an interior angle of a regular polygon with m sides is $\frac{2}{3}$ the measure of an interior angle of a regular polygon with n sides. Compute all ordered pairs (m,n).

ANSWERS:

7. 125
8. $\sqrt{58}$
9. 267
10. 49
11. 0
12. $(3,4),(4,8),(5,20)$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest THREE**

S01B13. Compute, in <u>inches</u>, the radius of a circle if a 10-foot chord is 4 feet from the center.

S01B14. Compute the number of 3-letter arrangements if the letters in each arrangement are in alphabetical order with no letter repeated.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01B15. If $x^2 + x - N = 0$, and N is chosen at random from all integers $1 \le N \le 100$, compute the probability that the equation will have integral roots.

S01B16. A triangle with integral sides has perimeter 8. Compute its area.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01B17. 10 married couples attended a party. Each man shook hands once with everyone except his wife, and no two women shook hands with each other. Compute how many handshakes took place.

S01B18. Compute the number of positive integers less than 50 with exactly four divisors.

<u>ANSWERS</u>:

13. $12\sqrt{41}$
14. 2600
15. $\frac{9}{100}$
16. $2\sqrt{2}$
17. 135
18. 15

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest FOUR**

S01B19. Compute $2^{2^{2^2}}$.

S01B20. The probability that Mike passes a test is $\frac{1}{3}$. The probability that Sue passes the test is $\frac{1}{4}$. The probability that Tom passes the test is $\frac{1}{5}$. Compute the probability that exactly two of them pass the test.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2001**

S01B21. A square and a regular hexagon have equal perimeters. The area of the hexagon is $54\sqrt{3}$. Compute the area of the square.

S01B22. All 5-digit numbers using the digits 1, 2, 3, 4, and 5 without repetition are put in numerical order from the smallest to the largest. Compute the 100^{th} number on this list.

PART III: 10 minutes **NYCIML Contest Four** **Spring 2001**

S01B23. All sides of a cube whose volume is 1000 cubic units are painted. The cube is then cut into cubes each of whose volume is 1 cubic unit. Compute the number of cubes with exactly two painted faces.

S01B24. Compute the number of 6-digit integers, the product of whose digits is exactly 1000.

<u>ANSWERS</u>:

19. 65,536
20. $\frac{3}{20}$
21. 81
22. 51,342
23. 96
24. 200

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest FIVE**

S01B25. In triangle ABC, $\angle C$ is a right angle. Compute $\cos A \cos B - \sin A \sin B$.

S01B26. $[x]$ means the greatest integer less than or equal to x. Compute x: $x[x] = 22$.

PART II: 10 minutes **NYCIML Contest Five** **Spring 2001**

S01B27. A circle is inscribed in an equilateral triangle with side 12. Compute the sum of the areas of the three regions inside the triangle and outside the circle.

S01B28. Compute $\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}$.

PART III: 10 minutes **NYCIML Contest Five** **Spring 2001**

S01B29. Compute the sum of the first 100 positive odd integers.

S01B30. A circle is circumscribed around a triangle with sides 8, 10, and 10. Compute the radius of the circle.

ANSWERS:

25. 0
26. -4.4
27. $36\sqrt{3}-12\pi$
28. $\dfrac{1+\sqrt{13}}{2}$
29. 10,000
30. $\dfrac{25\sqrt{21}}{21}$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **Spring 2001**

SOLUTIONS

S01B1. The units digit of the powers of 7 are in cycles of 4, (7,9,3,1). Since 2001 is one more than a multiple of 4, the units digit is 7.

S01B2. Let $x^{\frac{1}{3}} = y \to y + 6 = -\frac{8}{y} \to y^2 + 6y + 8 = 0$

$y = -4, y = -2 \to x \in \{-8, -64\}$

S01B3. Let x and y = 1. $(1+1)^8 = \mathbf{256}$

S01B4.

Factor: $\frac{2000!(2001-1)}{2000!((2002)(2001)-2)} = \frac{2000}{(2002)(2001)-2}$

Let $N = 2000 \to \frac{N}{(N+2)(N+1)-2} = \frac{N}{N(N+3)} = \frac{1}{N+3} = \frac{1}{\mathbf{2003}}$

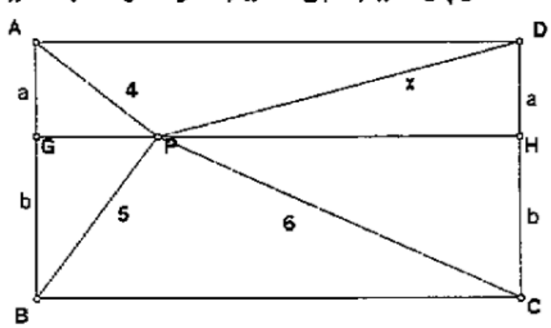
S01B5. The sum will be $\frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{1}{9}$

S01B6. Through point P, draw a line parallel to $\overline{AD}$.

$4^2 - a^2 = 5^2 - b^2$

$x^2 - a^2 = 6^2 - b^2$

$x^2 - 4^2 = 6^2 - 5^2 \to x^2 = 27 \to x = 3\sqrt{3}$



New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER TWO** **Spring 2001**

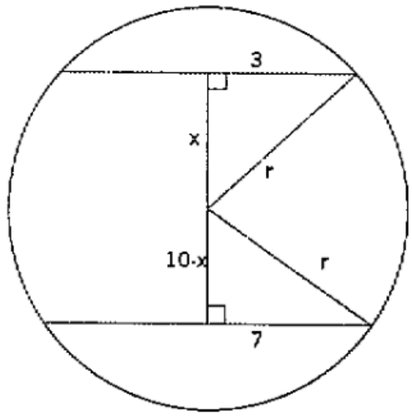
SOLUTIONS

S01B7. We have $\sqrt{5\cdot 5^5}=\sqrt{5^6}=5^3=\mathbf{125}$.

S01B8.

$$r^2=x^2+9=(10-x)^2+49=x^2-20x+149$$

$$x=7\rightarrow r=\sqrt{\mathbf{58}}$$



S01B9. There are 333 multiples of 3 less than 1000. 66 of these are multiples of 15, and therefore are multiples of 5. 333-66=**267**.

S01B10. The terms of a geometric are a, ar, ar^2, ar^3. Since $a=1, r^3=7$, the product of the terms is $a^4r^6=\mathbf{49}$. Or

the terms must be $7^0, 7^{\frac{1}{3}}, 7^{\frac{2}{3}}, 7$ and the product is $7^2=\mathbf{49}$.

S01B11. $3(a+2d)=6(a+5d)\rightarrow a=-8d\rightarrow a+(-a)=\mathbf{0}$.

S01B12.

$$\frac{(n-2)\cdot 180}{n}=\frac{3}{2}\cdot\frac{(m-2)\cdot 180}{m}$$

$n=\dfrac{4m}{6-m}$. Since $m\geq 3$, m can only be 3, 4, or 5.

$\{(3,4),(4,8),(5,20)\}$

Or, m must be less than 6 since if $m\geq 6$, the angles are greater than or equal to 120 and $\frac{3}{2}$ of this is greater than or equal to 180. Investigation shows that all three produce solutions.

New York City
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SENIOR B DIVISION **CONTEST NUMBER THREE** **Spring 2001**

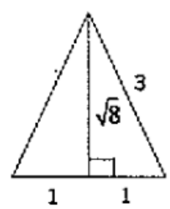
SOLUTIONS

S01B13. $r = \sqrt{4^2+5^2} = \sqrt{41}$ feet $= \mathbf{12\sqrt{41}}$ inches.

S01B14. Any three different letters will produce one arrangement in alphabetical order. ${}_{26}C_3 = \dfrac{26\cdot 25\cdot 24}{3\cdot 2\cdot 1} = \mathbf{2600}$.

S01B15. N must be in the form a(a+1), where a is an integer. Thus N may be 2,6,12,20,30,42,56,72,or 90. The probability is $\mathbf{\dfrac{9}{100}}$.

S01B16. The only triangle possible has sides 3,3,and 2. The area of this triangle is $\sqrt{8} = \mathbf{2\sqrt{2}}$.



S01B17. Each man shook hands with each woman other than his wife for 10(9) = 90 handshakes. Each man shook hands with each other for ${}_{10}C_2$ handshakes. $90 + {}_{10}C_2 = 90 + 45 =$ **135**.

S01B18. To have exactly 4 factors, a number must either be a perfect cube (except 1) or the product of two primes. 8,27,2(3),2(5),2(7), 2(11),2(13),2(17), 2(19), 2(23), 3(5),3(7), 3(11),3(13),5(7) for a total of **15** positive integers.

New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Spring 2001**

SOLUTIONS

S01B19. $2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65{,}536.$

S01B20. The probability $= \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{5}+\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{4}{5}=\mathbf{\frac{3}{20}}$.

S01B21. The area of a regular hexagon $= \frac{3s^2\sqrt{3}}{2} = 54\sqrt{3}$. $s = 6$ so the perimeter = 36 and the side of the square = 9. Area of the square = **81**.

S01B22. The first 96 begin with 1,2,3, and 4. The next 4 are 51234, 51243, 51324, and **51,342**.

S01B23. Each of the 12 edges will have 8 cubes painted on 2 faces, since the corners will be painted on 3 faces. 12 x 8 = **96**.

S01B24. $1000 = 5^3 2^3$, thus 3 of the digits must be 5. 2^3 can be created by 1 x 1 x 8, 2 x 2 x 2, or 1 x 2 x 4. The first has $\frac{6!}{2!3!}$ possibilities and the second $\frac{6!}{3!3!}$ possibilities and the third $\frac{6!}{3!}$ possibilities. 60 + 20 + 120 = **200**.

New York City Interscholastic Mathematics League

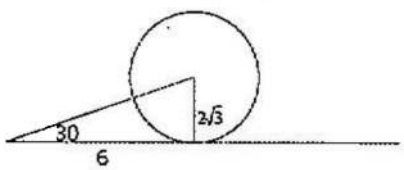
SENIOR B DIVISION **CONTEST NUMBER FIVE** **Spring 2001**

SOLUTIONS

S01B25. cosAcosB – sinAsinB = cos(A+B). Since A and B are complementary, cos(A+B) = cos90 = **0**.

S01B26. The number could either be between 4 and 5 or between –5 and –4. In the first case, $[x]=4$, but $4x=22\to x=5.5$, which lies outside the range. In the second case, $[x]=-5$, so $-5x=22\to x=\mathbf{-4.4}$

S01B27. The area of the triangle $=\frac{144}{4}\sqrt{3}=36\sqrt{3}$. The area of the circle = $\pi\left(2\sqrt{3}\right)^2=12\pi$. Thus the needed area is $\mathbf{36\sqrt{3}-12\pi}$.



S01B28. $\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}=x$

$=\sqrt{3+x}$ So $x^2=x+3\to x^2-x-3=0\to x=\frac{1\pm\sqrt{13}}{2}$, but since x must be positive, $x=\frac{1+\sqrt{13}}{2}$.

S01B29. This is an arithmetic progression so $S=\frac{100}{2}(1+199)=50(200)=10{,}000$.

Or the sum of the first n odd integers is $n^2\to100^2=10{,}000$.

S01B30.

$$4^2+(h-r)^2=r^2\to16+h^2-2hr+r^2=r^2$$

$$r=\frac{16+h^2}{2h},h=\sqrt{84}\to r=\frac{16+84}{2\sqrt{84}}=\frac{25}{\sqrt{21}}=\mathbf{\frac{25\sqrt{21}}{21}}.$$