New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION **CONTEST NUMBER ONE** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest One** **Spring 2001**

S01S1. A Tic-Toc-Tac-Toe board has 16 boxes arranged in 4 rows of 4 boxes. A player wins if he gets 4 boxes in a row (vertically, horizontally, or diagonally). If a mouse randomly chooses 4 boxes out of the 16, compute the probability that he wins.

S01S2. Compute $\dfrac{2001^2-1999^2+4000\times 4748}{2500^3+1500^3}$

PART II: 10 minutes **NYCIML Contest One** **Spring 2001**

S01S3. $ax+by+cz=3,\ bx+cy+az=30,\ cx+ay+bz=300$
If $a+b+c=9$, compute $x+y+z$.

S01S4. Compute $\dfrac{3}{2\times 5}+\dfrac{3}{5\times 8}+\dfrac{3}{8\times 11}+\dfrac{3}{11\times 14}+\ldots$

PART III: 10 minutes **NYCIML Contest One** **Spring 2001**

S01S5. $(x-2)$ is a factor of $x^2-(a+b)x+2a$ and $(b-1)x^2+ax+2a$. If a and b are integers, compute (a,b).

S01S6. A 2000 piece rectangular jigsaw puzzle has x pieces in each row and y pieces in each column. An interior piece is one that is completely surrounded on all sides by other pieces. Compute the maximum number of interior pieces.

ANSWERS

1. $\dfrac{1}{182}$
2. $\dfrac{1}{1000}$
3. 37
4. ½
5. (-1, 2)
6. 1824

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SENIOR A DIVISION | **CONTEST NUMBER TWO** | **SPRING 2001**

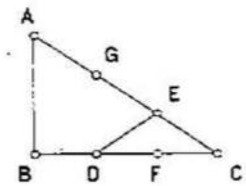
PART I: 10 minutes | **NYCIML Contest Two** | **Spring 2001**

S01S7. Compute the sum of the coefficients of the polynomial generated by the product of
$(x-2001)(x-2000)(x-1999)(x-1998)(x-1997)...(x-2)(x-1)$

S01S8. $x+y=a,\ xy=b,\ b\neq 0$. Express $\frac{1}{x^3}+\frac{1}{y^3}$ in simplest form, with no radicals, in terms of a and b.

PART II: 10 minutes | **NYCIML Contest Two** | **Spring 2001**

S01S9. In right triangle ABC, with B the right angle, $AB=12$, $BC=9$, $\overline{BC}$ is trisected by points D and F, and $\overline{AC}$ is trisected by points E and G, as shown. Compute the area of triangle DEC.



S01S10. The first term of an arithmetic sequence of n terms is n, the common difference is k and the sum of all the terms is $2000n$. Compute the smallest positive integral value of k.

PART III: 10 minutes | **NYCIML Contest Two** | **Spring 2001**

S01S11. The millennium race was held on New Year's Eve. The 2001 runners were randomly numbered from 1 to 2001, consecutively. Compute the probability that the numbers of the runners finishing first, second, and third, respectively, are in ascending order.

S01S12. If $\sin 5^\circ = a$, express $\sin 2000^\circ$ in terms of a in simplest form with no other trigonometric functions.

ANSWERS:

7. 0
8. $\frac{a^3-3ab}{b^3}$
9. 12
10. 1997
11. $\frac{1}{6}$
12. $(8a^3-4a)\sqrt{1-a^2}$ or $8a^3\sqrt{1-a^2}-4a\sqrt{1-a^2}$

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SENIOR A DIVISION **CONTEST NUMBER THREE** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01S13. The winning percentage for a team is determined by dividing the number of wins by the number of games played and rounding off to the nearest thousandth. If the Blue Sox have won 210 out of 550 games played, compute the minimum number of games they must win, from now on, to have a winning percentage of .400 or higher.

S01S14. Compute the number of distinct 3 letter arrangements that may be formed using the letters from the word "ALBANY". (Only an "A" can be used twice in an arrangement).

PART II: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01S15. A hexagon has an inscribed circle and a circumscribed circle. The area of the smaller circle is k times the area of the larger circle. Compute k.

S01S16. A certain diagnostic tool is correct 90% of the time. If a person tests negative using the tool, the test is performed a second time and only the new result is counted. If 30% of the actual population is positive, compute the percentage, to the nearest tenth, of the population that is properly diagnosed.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01S17. 1331 in base b is equivalent to 1000000 in base a. Compute the minimum value of $a + b$.

S01S18. $\log_a b = c$ and a, b, c are positive integers greater than 1. Compute the number of ordered pairs (a, c) such that b is a perfect cube less than or equal to 2001.

ANSWERS:

13. 17
14. 72
15. $\frac{3}{4}$
16. 86.4
17. 11
18. 16

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SENIOR A DIVISION **CONTEST NUMBER FOUR** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2001**

S01S19. Gabrielle wrote all the letters of the alphabet from A to Z. Next, she wrote every other letter of the alphabet, beginning with B. The third time, she wrote every third letter, beginning with C. She continued this pattern until the 26^{th} time, when she only wrote the letter Z. Compute the total number of vowels (a, e, i, o, u) written by Gabrielle.

S01S20. x is a real number and $\sqrt{4x^2-64x+400}=3y$. Compute the minimum value of y.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2001**

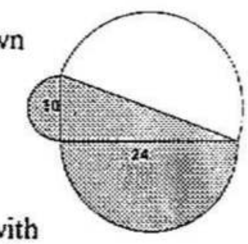
S01S21. There are $2001k$ diagonals in a 2001-sided convex polygon. Compute k.

S01S22. If $xy=2001$, compute

$$\frac{1}{x(x+y)}+\frac{1}{(x+y)(x+2y)}+\frac{1}{(x+2y)(x+3y)}+\frac{1}{(x+3y)(x+4y)}+\ldots$$

PART III: 10 minutes **NYCIML Contest Four** **Spring 2001**

S01S23. The legs of a right triangle are 10 and 24. Three semicircles are drawn using the sides of the triangle as diameters, as shown. The sum of the areas of the semicircles on the legs and the area of the triangle, minus the area of the semicircle on the hypotenuse may be expressed as $a+b\pi$. Compute (a,b).



S01S24. A basketball coach has 8 identical basketballs and a basketball rack with three different shelves. Each shelf can hold a maximum of 4 basketballs. Compute the number of distinct ways in which the balls can be placed on the rack.

ANSWERS:

19. 14
20. 4
21. 999
22. $\frac{1}{2001}$
23. (120,0)
24. 15

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SENIOR A DIVISION	**CONTEST NUMBER FIVE**	**SPRING 2001**
PART I: 10 minutes	**NYCIML Contest Five**	**Spring 2001**

S01S25. A fox is chasing a rabbit along a straight path. The rabbit is 2000 feet ahead of the fox. Each second, the fox advances 5 feet towards the rabbit and then the rabbit runs 3 feet away from the fox. Compute the number of seconds needed for the fox to catch the rabbit.

S01S26. Compute all x such that $|x|+|x-3|=3$.

PART II: 10 minutes **NYCIML Contest Five** **Spring 2001**

S01S27. A tic-tac-toe board has 9 boxes arranged in 3 rows of 3 boxes. A player wins if he gets 3 boxes in a row (vertically, horizontally, or diagonally). A goose randomly chooses 5 boxes out of the 9. Compute the probability that the goose wins.

S01S28. The graph of $y=ax^2+bx+c$ is a parabola. The points (2, 135), (-1, 57), and (-2, 67) are on the parabola. Compute $a+b+c$.

PART III: 10 minutes **NYCIML Contest Five** **Spring 2001**

S01S29. Aaron has x blocks. He realized that when 3, 4, or 5 friends visited, he could divide the blocks up equally among his friends and himself. One day, 7 friends came over. When they tried to divide the blocks up, there was one block left over. Compute the smallest possible value of x.

S01S30. A regular n-sided polygon has an inscribed circle and a circumscribed circle. The area of the smaller circle is k times the area of the larger circle. Express k in terms of n.

Answers:

25. 999
26. $0 \le x \le 3$
27. $\frac{49}{63}$
28. 91
29. 120
30. $k=\cos^2\left(\frac{\pi}{n}\right)$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION **SPRING, 2001** **SOLUTIONS** | **CONTEST NUMBER ONE**

S01S1 **Answer:** 1 / 182 . There are ${}_{16}C_4$ ways to choose 4 boxes. To win you must place the winning 4 boxes vertically (4 ways), horizontally (4 ways) or diagonally (2 ways). Therefore the probability is $10 / {}_{16}C_4 = 1 / 182$

S01S2 **Answer:** 1 / 1000 or .001 . Use the formulas for the difference between two squares and the sum of two cubes to factor the numerator and denominator .

$$\frac{2001^2 - 1999^2 + 4000(4748)}{2500^3 + 1500^3} = \frac{(2001-1999)(2001+1999) + 4000(4748)}{(2500+1500)(2500^2 - 2500(1500) + 1500^2)}$$

$$\frac{2(4000) + 4748(4000)}{(4000)(10000)(625 - 375 + 225)} = \frac{4750}{10000(475)} = \frac{1}{1000}$$

S01S3 **Answer:** 37. By adding the three equations

$$\begin{aligned} ax + by + cz &= 3 \\ bx + cy + az &= 30 \\ cx + ay + bz &= 300 \end{aligned}$$

we get $(ax + bx + cx) + (ay + by + cy) + (az + bz + cz) = 333$

$(a + b + c)(x + y + z) = 333$; $9(x + y + z) = 333$ $\therefore$ $(x + y + z) = 333/9$

S01S4 **Answer:** ½ . The series may be rewritten $\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{14}\right)+\ldots=\frac{1}{2}$

S01S5 **Answer:** (-1, 2). Since $x - 2$ is a factor of both expressions, then substituting 2 for x in either one yields 0.

$x^2 - (a+b)x + 2a$ and $(b-1)x^2 + ax + 2a$.

$4 - 2(a+b) + 2a = 0$ and $4(b-1) + 2a + 2a = 0$

$4 - 2b = 0$ or $b = 2$ and $4(2-1) + 2a + 2a = 0$; $a = -1$.

S01S6 **Answer:** 1824. x*y = 2000. The number of exterior pieces is 2x + 2y- 4. The number of interior pieces is 2000 - (2x +2y -4) or 2004 - 2(x + y) whose maximum occurs when x+y is a minimum. Since x and y must be two integers with product of 2000, therefore x and y are 40 and 50. The minimum number of interior pieces is 2004 - 2 *90 = 1824 .

S01S7 **Answer:** 0. The sum of the coefficients of a polynomial may be determined by substituting 1 for x because each term equals the value of its coefficient.
So the sum
$= (1 - 2001)(1 - 2000)(1 - 1999)(1 - 1998)(1 - 1997) \ldots (1 - 2)(1 - 1) = 0$.

S01S8 **Answer:** $(a^3 - 3ab)/b^3$.
$$\frac{1}{x^3} + \frac{1}{y^3} = \frac{x^3 + y^3}{x^3y^3} = \frac{(x+y)^3 - 3x^2y + 3xy^2}{b^3} = \frac{a^3 - 3(xy)(x+y)}{b^3}$$
or $(a^3 - 3ab)/b^3$.

S01S9 **Answer:** 12. The area of $\triangle ABC = \frac{1}{2}(9)(12) = 54$.
Using Area $= \frac{1}{2}ab \sin C = \frac{1}{2}(9)(15)\sin C = 54$ and $\sin C = 4/5$.
The area of $\triangle DEC = \frac{1}{2}(EC)(DC)\sin C = \frac{1}{2}(5)(6)(4/5) = 12$.

S01S10 **Answer:** 1997. $2000n = n + (n+k) + (n+2k) + (n+3k) + \ldots + (n + (n-1)k)$
$2000n = n * n + (k + 2k + 3k + \ldots + (n-1)k) = n^2 + (k(n-1)n/2)$; dividing by n we get $2000 = n + (k(n-1)/2)$ or $4000 = 2n + k(n-1)$ $\therefore$ $n = (4000 + k)/(k+2)$
$$= 1 + \frac{3998}{k+2}$$
Since n is an integer, 3998 is divisible by $k+2$ so $k+2 = 2$ or 1999, but $k \neq 0$ $\therefore$ $k =$ 1997.

S01S11 **Answer:** 1 / 6. The numbers of the three runners finishing 1st, 2nd and 3rd are irrelevant. There are 6 ways to write three numbers in any arbitrary order and only one of these six is in ascending order.

S01S12 **Answer:** $\left(8a^3 - 4a\right)\sqrt{1-a^2}$. Sin 2000° = sin 200° = - sin 20°
Using the double angle formula $\sin(2x) = 2\sin x \cos x$ and $\cos 2x = 1 - 2\sin^2 x$,
$-\sin 20° = -2\sin 10° \cos 10° = -2(2\sin 5° \cos 5°)(1 - 2\sin^2 5°) = \left(8a^3 - 4a\right)\sqrt{1-a^2}$.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION SPRING, 2001 SOLUTIONS | **CONTEST NUMBER THREE**

S01S13 **Answer:** 17 . If the Blue Sox win the next x games, the team winning percentage would be $(210+x)/(550+x)$.
$\frac{210+x}{550+x} \geq .400 = \frac{2}{5}$ Cross multiply and solve for x. $x \geq 50/3 \therefore x = 17$.

S01S14 **Answer:** 72. There are two possibilities: either each of the three letters is distinct or there are two "A"'s and another letter.
If the letters are different there are ${}_5C_3 = 10$ ways to select the letters and 3! ways to arrange each, for a total of 60 arrangements. If there are two "A"'s there are 3 ways to arrange each and ${}_4C_1 = 4$ ways to select the letter, for a total of 12 arrangements. $\therefore$ there are 72 arrangements.

S01S15 **Answer:** 3 / 4 . If the hexagon had a side of length 2 (there is no loss of generality), then the circumscribed circle would have a radius of 2. The inscribed circle would have a radius of $\sqrt{3}$. The area of the larger circle would be 4π and the area of the smaller circle would be 3π. $A_{smaller} / A_{larger} = 3/4$.

S01S16 **Answer:** 86.4. There are three groups that are properly diagnosed. Group #1 tested positive on the first try and they are positive. Group #2 tested negative on the first try but positive on the second try and they are positive. And Group # 3 tested negative both times and they are negative.
Group #1 is properly diagnosed (90%) (30%) = 27 %
Group #2 is properly diagnosed (10%) (30%)(90%) = 2.7 %
Group #3 is properly diagnosed (90%) (70%)(90%) = 56.7 %
$\therefore$ the probability that the person was properly diagnosed was 86.4 %.

S01S17 **Answer:** 11. In base 10 the equation becomes $b^3 + 3b^2 + 3b + 1 = a^6$ (where $b > 3$). Or $(b+1)^3 = a^6$. So $b + 1 = a^2$. The first values for a and b that satisfy this equation are $a = 3$ and $b = 8$.

S01S18 **Answer:** 16. Since b is a perfect cube and $a^c = b$, c is a multiple of 3 or a is a perfect cube, or both. If $c = 3$, $2 \leq a \leq 12$, since $b \leq 2001$. This gives 11 solutions. If $c = 6$, $a = 2$ or 3, giving 2 more solutions. If $c = 9$, $a = 2$, giving 1 more solution. Clearly, c cannot be 12 or greater. If $a = 8$, $c = 2$ or 3, but we have already considered the case where $c = 3$, so we have 1 new solution. If $a = 27$ (the greatest possible cube a can be), $c = 2$, giving 1 new solution. There are 16 total solutions.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION SPRING, 2001 SOLUTIONS CONTEST NUMBER FOUR

S01S19 **Answer:** 14. The number of times a letter is written is equal to the number of divisors of when it occurs. "A" is the first letter, it only appears once (one has only divisor 1). The letter "E" is the fifth letter and 5 has two divisors (1 and 5).The letter "I" is the ninth letter. Nine has three factors (1, 3, and 9), so it appears three times. "O" is the 15th letter and appears 4 times (15 ' s divisors are 1, 3, 5, and 15). "U" is the 21st letter and appears 4 times (1, 3, 7, and 21). The total number of times "A", "E","I","O", and "U" appear is $1 + 2 + 3 + 4 + 4 = 14$.

S01S20 **Answer:** 4 . By completing the square the radicand becomes $(2x-16)^2 + 144$
The minimum value of the radicand (and of $3y$) is when $2x-16 = 0$ $\therefore$ $3y = \sqrt{144}$
Or $y = 4$.

S01S21 **Answer:** 999. There are 2001 vertices each of which can only be connected to 1998 other vertices to form diagonals, so there are $2001 * 1998 / 2$ ways.
Therefore $k = 999$.

S01S22 **Answer:** 1 / 2001. The expression may be written as follows

$$\frac{1}{y}\left(\frac{1}{x}-\frac{1}{x+y}\right)+\frac{1}{y}\left(\frac{1}{x+y}-\frac{1}{x+2y}\right)+\frac{1}{y}\left(\frac{1}{x+2y}-\frac{1}{x+3y}\right)+\ldots=\frac{1}{y}\times\frac{1}{x}=\frac{1}{xy}=\frac{1}{2001}$$

S01S23 **Answer:** (120, 0). By the Pythagorean Theorem, we see that the area of the two smaller semicircles is equal to the area of the larger semicircle. Therefore the area we want is the area of the right triangle; $A = \frac{1}{2} b h = \frac{1}{2} (10)(24) = 120$.

S01S24 **Answer:** 15. With 8 balls, 3 shelves and a maximum of 4 balls on each shelf, we can make a list of all possible ways of placing the balls on the shelves. {(4,4,0), (4,3,1), (4,2,2), (4,1,3), (4,0,4), (3,4,1), (3,3,2), (3,2,3), (3,1,4), (2,4,2), (2,3,3), (2,2,4), (1,4,3), (1,3,4), (0,4,4)}

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION SPRING, 2001 SOLUTIONS CONTEST NUMBER FIVE

S01S25 **Answer:** 999. The rabbit has a 2000 foot head start. With each second the fox gains 2 feet. After 998 seconds, the rabbit's lead is only 4 feet (2000 - 2*998). With the next jump the fox gains the 5 feet and will catch the rabbit before it can get away.

S01S26 **Answer:** $0 \le x \le 3$. We must consider three intervals; $x < 0$, $0 \le x \le 3$, $x > 3$.
If $x < 0$, the equation becomes $-x + -(x-3) = 3$ or $x = 0$.
If $0 \le x \le 3$, the equation becomes $x + -(x-3) = 3$ or $3 = 3$ which is always true.
If $x > 3$, the equation becomes $x + (x-3) = 3$ or $x = 3$ which is a contradiction.
$\therefore 0 \le x \le 3$

S01S27 **Answer:** 49 / 63. The numerator is the number of ways to win in Tic-Tac-Toe. There are 8 ways to win with 3 boxes leaving 6 boxes to choose two ($8 * {}_6C_2 = 120$). However, the overlaps must be removed. The winners that have 3 across or 3 down have 5 overlaps each (30 total). The diagonals have 7 overlaps (14 total). Therefore we must remove half the overlaps (44/ 2 =22) that leaves us with 120- 22= 98 possible ways to win. The denominator is the number of ways 5 boxes may be chosen from 9 boxes or ${}_9C_5 = 126$.
98 / 126 = 49 /63

S01S28 **Answer:** 91. By substituting the three points into $y = ax^2 + bx + c$, we get
$135 = 4a + 2b + c$ $57 = a - b + c$ $67 = 4a - 2b + c$
Solving the equations simultaneously, we get $b = 17$, $a = 9$ and $c = 65$.
Therefore $a + b + c = 91$.

S01S29 **Answer:** 120. Because Aaron was able to divide the blocks evenly among 3, 4 or 5 people, the number of blocks must be divisible by 60. Also, the remainder must be one when divided by 7. If we allow the number of blocks to be 60n, then 60n / 7 must have a remainder of 1. Therefore the lowest integral value of n is 2 so there must be 120 blocks.

S01S30 **Answer:** $k = \cos^2\left(\frac{\pi}{n}\right)$. If you look at one side of the n-sided polygon, the ratio becomes apparent. In the triangle, the altitude is the radius of the inner circle and the hypotenuse is the radius of the outer circle. The ratio of the smaller radius to the larger radius is $\cos\left(\frac{\pi}{n}\right)$. Therefore their areas are proportional to the ratio of the radii squared or $\cos^2\left(\frac{\pi}{n}\right)$.