New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **SPRING 2001**

PART I: 10 minutes | **NYCIML Contest One** | **Spring 2001**

S01J1. A ½ mile long train, traveling at a constant speed of 30 miles per hour, enters a 3-mile tunnel at 4:00 PM. Compute the earliest time after 4:00 PM that no part of the train is in the tunnel.

S01J2. Compute the units digit of $2003^{2003^{2003}}$.

PART II: 10 minutes | **NYCIML Contest One** | **Spring 2001**

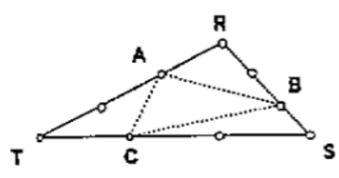
S01J3. Compute the length of the radius of a circle inscribed in a right triangle with hypotenuse of length 26 and one leg of length 10.

S01J4. Stanley has 7 rectangular solid blocks of dimensions 1 inch by 4 inches by 9 inches. He stacks the blocks by placing one block on top of another to make a tower 7 blocks high. He can turn the blocks so that each block can contribute 1 inch, 4 inches or 9 inches to the height of the tower. Compute the number of different heights of the tower that can be achieved.

PART III: 10 minutes | **NYCIML Contest One** | **Spring 2001**

S01J5. Compute all integers n such that $\frac{2n}{n-1}$ is an integer.

S01J6. Each side of triangle RST is trisected and triangle ABC is formed by joining alternate trisection points as in the diagram. The area of triangle RST is 84. Compute the area of the triangle ABC.



ANSWERS:
J1. 4:07 p.m.
J2. 7
J3. 4
J4. 36
J5. -1,0,2,3
J6. 28

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JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2001**
PART I: 10 minutes **NYCIML Contest Two** **Spring 2001**

S01J7. $a=\frac{\sqrt{5}-1}{3\sqrt{3}}, b=\frac{\sqrt{5}+1}{3\sqrt{3}}, c=\frac{\sqrt{5}}{3}$. Compute the value of $\frac{a^2+b^2-c^2}{2ab}$.

S01J8. A two-digit prime number P is 21 more than the sum of the squares of its digits. Compute all possible values of P.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2001**

S01J9. Among 100 students in the senior class, all had studied at least one of the following languages: Polish, Romanian, and Chinese. 6 students studied all three languages. 48 students studied Polish, 54 students studied Romanian, and 52 studied Chinese. Compute how many students studied exactly two languages.

S01J10. A rectangle with area 200π has two congruent circles inscribed in it, each circle tangent to three sides of the rectangle. If the area of the region of the circles' overlap equals the sum of the areas of the parts of the rectangle exterior to both circles, compute the radius of one of the circles.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2001**

S01J11. $f(x)=x^7+x^3+2x+|x|-1$. Compute the value of $f(2001)+f(-2001)$.

S01J12. A fair coin is flipped 17 times. The probability of at least 11 heads in a row is $\frac{a}{2^b}$, where a is an odd integer, and b is an integer. Compute (a,b).

ANSWERS:

J7. $-\frac{3}{8}$
J8. 31,71
J9. 42
J10. 10
J11. 4000
J12. (1,9)

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JUNIOR DIVISION **CONTEST NUMBER THREE** **SPRING 2001**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01J13. A number is a palindrome if it is a positive integer that reads the same backwards and forwards. For example, 58785 is a palindrome. Compute the number of 5 digit palindromes that have the sum of the digits less than 7.

S01J14. Beth can paint a room by herself in 6 hours, Jay can paint the same room by himself in 9 hours, and Dina can paint the room by herself in 12 hours. One hour after Beth begins painting the room, Jay begins helping her, and one hour after that, Dina begins helping them. Compute the number of hours it will take to paint the room from the time that Beth began.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2001**

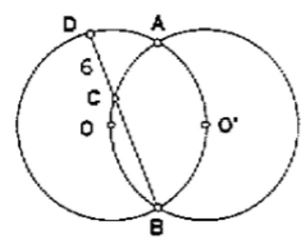
S01J15. If two factors of $2x^3 - ax + b$ are $x+2$ and $x-1$, compute $|a|+|b|$.

S01J16. Two cards are chosen without replacement from a standard deck of 52 cards. Compute the probability that at least one of the cards is a queen AND at least one of the cards is a heart.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2001**

S01J17. Compute all ordered triples of integers (a,b,c) such that $a^2+b^2+c^2=110$ and $0<a<b<c$.

S01J18. Circles O and O' each pass through the center of the other and intersect at A and B. A line segment from B intersects the circles at C and D, as shown in the diagram. If $CD = 6$, compute the area of triangle CDA.



ANSWERS:

J13. 14

J14. $\frac{46}{13}$

J15. 10

J16. $\frac{29}{442}$

J17. (5,6,7), (1,3,10), (2,5,9)

J18. $9\sqrt{3}$

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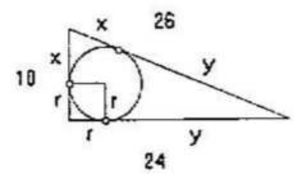
JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2001**

SOLUTIONS

S01J1. 30 miles per hour is equivalent to ½ mile per minute. Since the train has to go 3½ miles to completely clear the tunnel, the answer is **4:07 pm.**

S01J2. The units digits of sequential powers of 2003 form a cycle $3 \to 9 \to 7 \to 1$ of length 4. Since the exponent, $2003^{2003} \equiv (-1)^{2003} \equiv -1 \equiv 3 (\text{mod } 4)$, we want the third term in the cycle, or **7**.

S01J3. The other leg has length 24. Assigning *x*, *y*, and *r* as in the picture, we have $x + r = 10$, $y + r = 24$, and $x + y = 26$.



$$r = \frac{(x+r)+(y+r)-(x+y)}{2} = \frac{10+24-26}{2} = 4$$

OR

The radius of a circle inscribed in a triangle (the inradius) is the area of the triangle divided by the semiperimeter of the triangle. $\dfrac{120}{\frac{1}{2} \cdot 60} = 4$

S01J4. In a stack of 7 blocks, of height *h*, there will be *x* stacked with height 1', *y* stacked with height 4', and *z* stacked with height 9'. $x + y + z = 7$, $x + 4y + 9z = h$, $3y + 8z = h - 7$, and since $7 \le h \le 63 \to 0 \le 3y + 8z \le 56$. (Or, the minimum height of the blocks is 7, and the maximum is 63, thus there are 57 possible values for the height of the stack). For $y = 0$ there are 8 possible values for *z*. For $y = 1$, there are 7 possible values for *z*. For $y = 2$, there are 6 possible values for *z*, etc. to $y = 7$, there is 1 possible value for *z*. This gives **36** heights the stack can take.

S01J5. $\dfrac{2n}{n-1} = \dfrac{2n-2+2}{n-1} = 2 + \dfrac{2}{n-1}$, so $n-1$ can be –2, -1, 1, or 2, making the solution set $\{-1, 0, 2, 3\}$

S01J6. The length of base $\overline{CS}$ of triangle *BCS* is 2/3 the length of base $\overline{TS}$ of triangle *RTS*, and the length of the height of triangle *BCS* is 1/3 the length of the height of triangle *RTS*, so $\text{Area}(\Delta BCS) = \frac{2}{9}\text{Area}(\Delta RTS)$. Analogously,

$$\text{Area}(\Delta CAT) = \text{Area}(\Delta ABR) = \frac{2}{9}\text{Area}(\Delta RTS).$$

$$\text{Area}(\Delta BCS) + \text{Area}(\Delta CAT) + \text{Area}(\Delta ABR) = \frac{2}{3}\text{Area}(RTS).$$

$$\text{Area}(\Delta ABC) = \text{Area}(\Delta RTS) - \left(\text{Area}(\Delta BCS) + \text{Area}(\Delta CAT) + \text{Area}(\Delta ABR)\right) = \frac{1}{3}\text{Area}(\Delta RTS) = \mathbf{28}$$

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JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2001**

SOLUTIONS

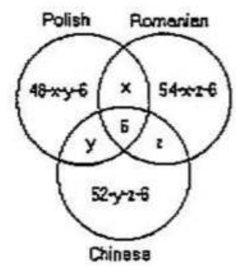
S01J7. $\frac{a^2+b^2-c^2}{2ab}=\frac{(a+b)^2-2ab-c^2}{2ab}=\frac{\frac{20}{27}-\frac{8}{27}-\frac{5}{9}}{\frac{8}{27}}=-\frac{3}{8}$.

S01J8. Let $P=10t+u \rightarrow 10t+u=t^2+u^2+21 \rightarrow t^2-10t+u^2-u=-21$. Completing the square, we get $(t-5)^2+\left(u-\frac{1}{2}\right)^2=4\frac{1}{4}$, and since u is an odd digit, $u=1$, so $t-5=\pm 2 \rightarrow t=3$ or $t=7$, yielding two solutions: **31, 71.**

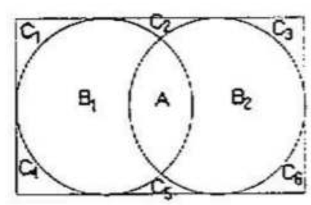
S01J9. Set up the Venn diagram as shown.

$48-x-y-6+54-x-z-6+52-y-z-6+x+y+z+6=100$

$x+y+z=\mathbf{42}$.



S01J10. Let A be the area of the region of overlap of the two circles. Let $B=B_1+B_2$ be the sum of the areas of the regions interior to exactly one of the circles. Let $C=C_1+C_2+C_3+C_4+C_5+C_6$ be the sum of the areas of the regions interior to the rectangle but exterior to the circles. $A+B+C$ equals the area of the rectangle. Since $A=C$, $A+B+C=2A+B=200\pi$ and, since $B_1=B_2$, $A+B_1=100\pi=\pi r^2 \rightarrow r=\mathbf{10}$.



S01J11. The first three terms will cancel, leaving $f(2001)+f(-2001)=|2001|-1+|-2001|-1=\mathbf{4000}$

S01J12.. Sequences of favorable flip results can be separated into seven distinct categories: H..H??????, TH..H?????, ?TH..H????, ??TH..H???, ???TH..H??, ????TH..H?, and ?????TH..H, (there are 11 H's in every "H..H",) according to where the 11 (or more) heads in a row are to start. This covers all favorable sequences. There are two possibilities for each "?", so the number of sequences in the first category is 2^6, and in each other category 2^5. This gives a total of $2^6+6 \cdot 2^5=2^8$ favorable sequences. Since there are 2^{17} possible outcomes for flipping a coin 17 times, the probability of obtaining a favorable outcome is $\frac{2^8}{2^{17}}=\frac{1}{2^9}$. $(a,b)=\mathbf{(1,9)}$

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JUNIOR DIVISION **CONTEST NUMBER THREE SOLUTIONS** **SPRING 2001**

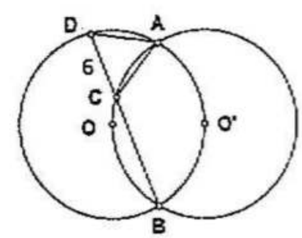
S01J13. The sum of the first two digits can be 1, 2, or 3. If it is 1, they have to be "10", and there are 5 possible values the middle digit can take. If it is 2, they can be "11" or "20", and there are 3 possible values the middle digit can take. If it is 3, they can be "12", "21", or "30", and the middle digit has to be zero. Adding all possible combinations we get $1\cdot 5+2\cdot 3+3\cdot 1=\mathbf{14}$.

S01J14. In 1 hour, Beth can do $\frac{1}{6}$ of the job, Jay can do $\frac{1}{9}$ of the job, and Dina can do $\frac{1}{12}$ of the job. Thus after the first two hours, $\frac{2}{6}+\frac{1}{9}=\frac{4}{9}$ of the job is completed, and $\frac{5}{9}$ remains. Since working together, the three can do $\frac{1}{6}+\frac{1}{9}+\frac{1}{12}=\frac{13}{36}$ of the job per hour; they will be done in $\frac{5}{9}\div\frac{13}{36}=\frac{20}{13}$ hours. The total time is $2+\frac{20}{13}=\mathbf{\frac{46}{13}}$ hours.

S01J15. Since -2 and 1 are roots of the equation $2x^3-ax+b=0$, $-16+2a+b=0$, and $2-a+b=0$. Solving, we get $\boldsymbol{a=6}$, $\boldsymbol{b=4}$, so $|a|+|b|=\mathbf{10}$.

S01J16. The probability that one of the cards is the queen of hearts is $\frac{1}{26}$. The probability that neither of the cards is the queen of hearts, but one card is a queen and the other a heart is $2\cdot\frac{3}{52}\cdot\frac{12}{51}=\frac{6}{221}$. Since these are mutually exclusive, the final probability is $\frac{1}{26}+\frac{6}{221}=\mathbf{\frac{29}{442}}$.

S01J17. The largest of the squares can be from 36 to 100. Trying one square is 100, the sum of the other two must be 10=1 + 9. Continuing like this, we find **(5,6,7), (1,3,10), (2,5,9).**



S01J18. AOO' and BOO' are equilateral triangles, so $m\widehat{AB}=120°$. Draw $\overline{AC}$ and $\overline{AD}$. $\angle ADB$ is inscribed in $\widehat{AB}$ in circle O, so $m\angle ADB=60°$. $\angle ACB$ is inscribed in major $\widehat{AB}$ in circle O', so $m\angle ACB=120°$, and $m\angle ACD=60°$. Therefore, ΔACD is equilateral and the area is $\frac{6^2}{4}\sqrt{3}=\mathbf{9\sqrt{3}}$