New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2000**

PART I: 10 minutes **NYCIML Contest One** **FALL 2000**

F2000SB1. In equilateral triangle ABC, $\overline{DE}$ is drawn parallel to $\overline{BC}$ so that D is on $\overline{AB}$ and E is on $\overline{AC}$. If $BC = 10$ and $DB = 2$, compute the area of trapezoid $DECB$.

F2000SB2. A fair die is rolled twice. Compute the probability that the result of the first roll exceeds the result of the second roll.

PART II: 10 minutes **NYCIML Contest One** **Fall 2000**

F2000SB3. A regular hexagon is inscribed in a circle. If the area of the hexagon is $54\sqrt{3}$, compute the area of the circle.

F2000SB4. If ${}_N C_1$, ${}_N C_2$ and ${}_N C_3$ form an arithmetic progression, compute N.

PART III: 10 minutes **NYCIML Contest One** **Fall 2000**

F2000SB5. The sum of all but one of the angles of a convex polygon is $3624°$. Compute the number of sides of the polygon.

F2000SB6. Three different integers are randomly chosen from $\{1, 2, 3, 4, 5, 6, 7\}$. Compute the probability that the sum of these three integers is odd.

ANSWERS:

1. $9\sqrt{3}$
2. $\frac{5}{12}$
3. 36π
4. 7
5. 23
6. $\frac{16}{35}$

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SENIOR B DIVISION	CONTEST NUMBER TWO	FALL 2000

PART I: 10 minutes **NYCIML Contest Two** **FALL 2000**

F2000SB7. Two successive increases of 25 % are equivalent to one increase of N %. Compute N.

F2000SB8. Compute N if $4\sqrt{N}-4=N$.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2000**

F2000SB9. Compute the area of an isosceles trapezoid with sides 6, 9, 9, 10.

F2000SB10. Compute the number of positive integral factors of 143,143.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2000**

F2000SB11. Compute the sum of all integers between 100 and 200 inclusive, that are divisible by 7.

F2000SB12. A game is won when a fair die is rolled three times and the third roll is the sum of the first two. If Sam wins the game, compute the probability that he rolled a 1 at least once.

ANSWERS:

7. 56.25
8. 4
9. $8\sqrt{77}$
10. 18
11. 2107
12. $\frac{3}{5}$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2000**

PART I: 10 minutes **NYCIML Contest Three** **FALL 2000**

F2000SB13. Compute the sum of the digits of all integers from 101 to 200 inclusive.

F2000SB14. Compute the number of ounces of water that must be added to 15 ounces of a 40 % acid solution to make it a 30 % acid solution.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2000**

F2000SB15. Compute the number of positive integers N, $N < 1000$, for which $\log_8 N$ is rational.

F2000SB16. Compute the sum $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{19\cdot 20}$.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2000**

F2000SB17. Compute all ordered pairs of integers (x, y) such that $x+y=xy$.

F2000SB18. The base of a pyramid is an equilateral triangle with sides of length 6. The lateral edges of the pyramid are of length 10. Compute the volume of the pyramid.

ANSWERS:

13. 1001
14. 5
15. 10
16. $\frac{19}{20}$
17. $(0,0),(2,2)$
18. $6\sqrt{66}$

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2000**

PART I: 10 minutes **NYCIML Contest Four** **FALL 2000**

F2000SB19. Compute the ordered pair (x, y) which is the solution of $x^2-8x+y^2+6y=-25$.

F2000SB20. If $\log 2=a$, express $\log\frac{5}{16}$ in terms of a, with no logarithms.

PART II: 10 minutes **NYCIML Contest Four** **Fall 2000**

F2000SB21. Compute r so that ${}_{12}C_r$ is a maximum.

F2000SB22. The vertices of triangle ABC are $A(-1,-2)$, $B(2,4)$, and $C(6,2)$. Compute the length of the altitude from B to $\overline{AC}$.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2000**

F2000SB23. Compute all values of x that are solutions of $3\cdot 9^{2x}-10\cdot 9^{x}+3=0$.

F2000SB24. Solve for all positive acute angles x: $2\csc x=\tan 72°+\tan 18°$.

ANSWERS:

19. (4, -3)
20. $1-5a$
21. 6
22. $\frac{6\sqrt{65}}{13}$
23. $\frac{1}{2}, -\frac{1}{2}$
24. 36

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2000**

PART I: 10 minutes **NYCIML Contest Five** **FALL 2000**

F2000SB25. If $i=\sqrt{-1}$, compute $(1+i)^{20}$.

F2000SB26. Working alone, John can paint a room in 8 hours. After he has been working for 3 hours, Dave helps him and together they complete the job in 2 more hours. Compute how long it would have taken Dave to paint the room alone.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2000**

F2000SB27. Compute, $\sin\left(Arc\sin\frac{3}{5}+Arc\cos\frac{5}{13}\right)$ where *Arc* sin and *Arc* cos denote principal value.

F2000SB28. Compute the number of ways four men and four women can be seated at a circular table if the men and women alternate around the table. (Two seatings are identical only when each person has the same right neighbor.)

PART III: 10 minutes **NYCIML Contest Five** **Fall 2000**

F2000SB29. Compute the length of a diagonal of a cube with edge of length 4.

F2000SB30. If 796, 1157, and 1594 are divided by the positive integer q, they all leave a positive remainder r. Compute r.

ANSWERS:

25. -1024

26. $5\frac{1}{3}$

27. $\frac{63}{65}$

28. 144

29. $4\sqrt{3}$

30. 17

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SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2000**

SOLUTIONS

F00SB1. $\text{Area}(\Delta ABC) = \frac{10^2}{4}\sqrt{3} = 25\sqrt{3}$ $\text{Area}(\Delta ADE) = 16\sqrt{3}$
$\text{Area}(DECB) = 25\sqrt{3} - 16\sqrt{3} = \mathbf{9\sqrt{3}}$

F00SB2 There are 36 possible results, of which 6 are the same result on both dice. 15 of the remaining 30 have the first roll exceeding the second roll.
$\frac{15}{36} = \mathbf{\frac{5}{12}}$.

F00SB3. The radius is equal to the side of the hexagon. $54\sqrt{3} = \frac{3}{2}s^2\sqrt{3}$ $s^2 = 36$
$s = 6$ $A = \mathbf{36\pi}$.

F00SB4 ${}_NC_2 - {}_NC_1 = {}_NC_3 - {}_NC_2$ $\frac{N(N-1)}{2} - N = \frac{N(N-1)(N-2)}{6} - \frac{N(N-1)}{2}$
$N^3 - 9N^2 + 14N = 0$ $N(N-2)(N-7) = 0$ $N = \mathbf{7}$.

OR

Construct a Pascal's triangle until the second, third, and forth terms of a row form an arithmetic progression. This occurs in the seventh row.

F00SB5. The sum of the angles is the next multiple of $180°$ greater than $3624°$. This is $3780°$. $3780° = (N-2)180°$ $N = \mathbf{23}$

F00SB6. Either all three integers are odd or one is odd and the other two are even.

$\frac{{}_4C_3 + 4 \cdot {}_3C_2}{{}_7C_3} = \frac{4 + 4 \cdot 3}{35} = \mathbf{\frac{16}{35}}$.

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SENIOR B DIVISION **CONTEST NUMBER TWO** **FALL 2000**

SOLUTIONS

F00SB7. $\frac{5}{4}\left(\frac{5}{4}x\right)=\frac{25}{16}x=1\frac{9}{16}x \quad \frac{9}{16}=56.25\%$.

F00SB8 $N=4\sqrt{N}-4 \quad N+4=4\sqrt{N} \quad (N+4)^2=16N \quad N^2-8N+16=0 \quad N=4$.

F00SB9. $A=\frac{1}{2}\sqrt{77}(6+10)=8\sqrt{77}$

F00SB10 $143143=143\cdot 1001=13\cdot 11\cdot 7\cdot 11\cdot 13=7^1 11^2 13^2$. Adding one to each exponent and multiplying, the number of factors is $2\cdot 3\cdot 3=\mathbf{18}$.

F00SB11. $7\cdot 15=105$ is the smallest. $7\cdot 28=196$ is the largest. There are 14 numbers. $S=\frac{14}{2}(105+196)=7(301)=\mathbf{2107}$.

F00SB12. There are 15 possibilities:

1 1 2	2 1 3	3 1 4	4 1 5	5 1 6
1 2 3	2 2 4	3 2 5	4 2 6	
1 3 4	2 3 5	3 3 6		
1 4 5	2 4 6			
1 5 6				

Of these 15 possibilities, 9 of them have 1 at least once. $\frac{9}{15}=\frac{3}{5}$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2000**

SOLUTIONS

F00SB13. There are 100 numbers. In the tens and units places, there are 10 of each digits. $10(0+1+2+3+\ldots+9)=450$. In the hundreds place, there are 99 ones and 1 two. $2(450)+99+2=\mathbf{1001}$.

F00SB14. There are .4(15) ounces of acid in the original solution. That equals .3(15+x) ounces of acid in the diluted solution. $.4(15)=.3(15+x)$ x=**5**.

F00SB15. $\log_8 N$ will be rational if N is any power of 2. There are **10** possible values: 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512.

F00SB16. $\frac{1}{N(N+1)}=\frac{1}{N}-\frac{1}{N+1}$ $S=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{19}-\frac{1}{20}$. All terms cancel out except for $\frac{1}{1}-\frac{1}{20}=\mathbf{\frac{19}{20}}$.

F00SB17. $xy-x-y+1=1$ $(x-1)(y-1)=1$ Either $x-1=y-1=-1$ or $x-1=y-1=1$. These give the solutions **(0,0)** and **(2,2)**.

F00SB18. $V=\frac{1}{3}Ah$. The area of the base = $\frac{6^2}{4}\sqrt{3}=9\sqrt{3}$. The height is one leg of a right triangle, which has hypotenuse of 10 and the other leg is the distance from the center of the equilateral triangle to the vertex of the triangle. This distance is $2\sqrt{3}$, since it is $\frac{2}{3}$ the altitude of an equilateral triangle with side 6.

$(2\sqrt{3})^2+h^2=10^2 \rightarrow h^2=88 \rightarrow h=\sqrt{88}$.

$V=\frac{1}{3}(9\sqrt{3})(\sqrt{88}) \rightarrow V=\mathbf{6\sqrt{66}}$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2000**

SOLUTIONS

F00SB19. $x^2-8x+16+y^2+6y+9=-25+25=0$. $(x-4)^2+(y+3)^2=0$. $(4,-3)$

F00SB20.

$$\log\frac{5}{16}=\log 5-\log 16=\log\frac{10}{2}-\log 2^4=\log 10-\log 2-4\log 2=1-a-4a=1\text{-}5a$$

F00SB21. As we can see by examining the middle number of any even power row in Pascal's triangle, the maximum value will be when r= ½ n. Therefore, r=**6**.

F00SB22. Since $AB=\sqrt{45}, BC=\sqrt{20}, AC=\sqrt{65}$, the triangle is a right triangle with hypotenuse $\overline{AC}$. The area is

$$\frac{1}{2}\sqrt{45}\sqrt{20}=15\rightarrow\frac{1}{2}h\sqrt{65}=15\rightarrow h=\frac{30}{\sqrt{65}}=\frac{30\sqrt{65}}{65}=\mathbf{\frac{6\sqrt{65}}{13}}$$

F00SB23.

$$\text{Let } y=9^x\Rightarrow 3\cdot y^2-10y+3=0\Rightarrow(3y-1)(y-3)=0\Rightarrow y=\frac{1}{3}\text{ or } y=3\Rightarrow$$

$$9^x=\frac{1}{3}\text{ or } 9^x=3\Rightarrow 3^{2x}=3^{-1}\text{ or } 3^{2x}=3^1\Rightarrow x\in\left\{\frac{1}{2},-\frac{1}{2}\right\}$$

F00SB24.

$$\frac{2}{\sin x}=\frac{\sin 72}{\cos 72}+\frac{\sin 18}{\cos 18}=$$

$$\frac{\sin 72\cos 18+\sin 18\cos 72}{\cos 72\cos 18}=\frac{\sin 72\cos 18+\sin 18\cos 72}{\sin 18\cos 18}$$

$$=\frac{\sin 90}{\frac{1}{2}\sin 36}=\frac{2}{\sin 36}=2\csc 36\rightarrow x=\mathbf{36}.$$

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2000**

SOLUTIONS

F00SB25. $(i+1)^2 = 1+2i+i^2 = 2i$

$(i+1)^{20} = (2i)^{10} = 2^{10}i^{10} = 1024(-1) = \textbf{-1024}$.

F00SB26. John completes $\frac{5}{8}$ of the job and Dave completes $\frac{2}{x}$ of the job.

$\frac{5}{8}+\frac{2}{x}=1 \rightarrow x = \mathbf{5\frac{1}{3}}$.

F00SB27.

$\sin\left(Arc\sin\frac{3}{5}+Arc\cos\frac{5}{13}\right) = \sin Arc\sin\frac{3}{5}\cos Arc\cos\frac{5}{13}+\sin Arc\cos\frac{5}{13}\cos Arc\sin\frac{3}{5} =$

$\frac{3}{5}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{4}{5}=\mathbf{\frac{63}{65}}$.

F00SB28. Seat the men first. Once the first man sits down, there are 3! ways the other men can sit. There are now 4! ways that the women can sit. (3!)(4!)= **144**.

F00SB29. The diagonal of a cube is the hypotenuse of a right triangle whose legs are an edge of the cube and the diagonal of a side.

$4^2+\left(4\sqrt{2}\right)^2 = x^2 \rightarrow x = \mathbf{4\sqrt{3}}$

F00SB30. since all the divisions leave the same remainder, q must divide the difference between any two numbers. 1157-796 = 361= 19(19). 1594-1157=437=19(23). Therefore, q=19 and dividing any of the numbers by 19, we get **r=17**.