NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER ONE**

PART I — ***FALL, 2000*** — ***CONTEST 1*** — ***TIME: 10 MINUTES***

F00S1 On a number line, an integer, N, and its square are 56 units apart. Find all values of N.

F00S2 In a geometric progression of constantly increasing values, the difference between the second and third terms is 12 and the difference between the fourth and fifth terms is 27. Compute the first term.

PART II — ***FALL, 2000*** — ***CONTEST 1*** — ***TIME: 10 MINUTES***

F00S3 The sum of all but one angle of a convex polygon is 2400°. Compute the measure of that one angle.

F00S4 How many positive integers less than or equal to 1000 are ***not*** multiples of 6 nor of 9?

PART III — ***FALL, 2000*** — ***CONTEST 1*** — ***TIME: 10 MINUTES***

F00S5 Amanda has 5 times as many brothers as sisters. Her brother Jason has twice as many brothers as sisters. How many children does this family have?

F00S6 In ΔABC, $AB = 18$, $BC = 15$ and $AC = 12$. If $m\angle B = x$, express $m\angle C$ in terms of x, in simplest form and with no inverse trigonometric functions in the answer.

ANSWERS:		
	F00S1	8, –7 (both required)
	F00S2	16
	F00S3	120°
	F00S4	778
	F00S5	7
	F00S6	$2x$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER TWO**

PART I *FALL, 2000* *CONTEST 2* *TIME: 10 MINUTES*

F00S7 If x is a positive integer greater than 2000, list the following in order from smallest to largest value:

$$2^{5x},\ 3^{4x},\ 4^{3x},\ 5^{2x}$$

F00S8 If $\sqrt{12+8\sqrt{2}}+\sqrt{12-8\sqrt{2}}=2\sqrt{N}$, compute N.

PART II *FALL, 2000* *CONTEST 2* *TIME: 10 MINUTES*

F00S9 A small regular hexagon is inscribed in a circle which is inscribed in a larger regular hexagon whose perimeter is 24. Compute the perimeter of the small hexagon.

F00S10 The base 6 representation of N is 100. If N! is multiplied out and written in base 6, how many terminal zeros (zeros after the last nonzero digit) are written?

PART III *FALL, 2000* *CONTEST 2* *TIME: 10 MINUTES*

F00S11 The sides of a triangle have lengths 4, 13, and 15. Compute the length of its longest altitude.

F00S12 If 1 is added to the product of four consecutive integers, the square root of the result is 55. If the smallest of the consecutive integers is n, compute both values of n.

ANSWERS:		
	F00S7	$5^{2x},\ 2^{5x},\ 4^{3x},\ 3^{4x}$
	F00S8	8
	F00S9	$12\sqrt{3}$
	F00S10	17
	F00S11	12
	F00S12	6, –9

New York City Interscholastic Math League

Senior A Division | **Contest Number Three**

Part I *Fall, 2000* *Contest 3* *Time: 10 Minutes*

F00S13 In a sequence of numbers, each term is formed by squaring each digit of the previous term and then adding the results. The first term of the sequence is 2000. Compute the 100^{th} term.

F00S14 If $3a + 7b + c = 320$ and $2a + 4b + c = 205$, compute $a + b + c$.

Part II *Fall, 2000* *Contest 3* *Time: 10 Minutes*

F00S15 Ashley answers all 100 questions in a math contest. She receives 7 points for each correct answer but loses 3 points for each wrong answer. If she scores a total of 330 points, how many questions did she get wrong?

F00S16 The sides of a triangle are 13, 14, and 15. A square is inscribed in the triangle so that two of its vertices are on side 14. Compute the length of one side of the square.

Part III *Fall, 2000* *Contest 3* *Time: 10 Minutes*

F00S17 Compute the least positive integer n for which $n^2 + 6n + 32$ is a multiple of 23.

F00S18 Compute the area of a triangle whose sides are 7, 13, and $5\sqrt{2}$.

Answers:		
	F00S13	37
	F00S14	90
	F00S15	37
	F00S16	$\frac{84}{13}$
	F00S17	20
	F00S18	17.5 or $\frac{35}{2}$ or $17\frac{1}{2}$

New York City Interscholastic Math League

Senior A Division | **Contest Number Four**

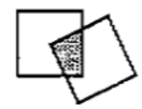
Part I *Fall, 2000* *Contest 4* *Time: 10 Minutes*

F00S19 For what value of A is the five-digit number 17,5AA exactly divisible by 24?

F00S20 In a numerical sequence, $f(n)$ is the n^{th} term. If $f(1) = 0$ and $f(n) = f(n-1) + n$ for $n > 1$, then $f(n) = an^2 + bn + c$. Compute the ordered triple (a, b, c).

Part II *Fall, 2000* *Contest 4* *Time: 10 Minutes*

F00S21 Two congruent squares each have an area of 100. A vertex of one is also the center of the other, as shown. Compute the area of the region common to both squares.



F00S22 Compute $\sum_{K=0}^{360^\circ} (\sin K + \cos K)$.

Part III *Fall, 2000* *Contest 4* *Time: 10 Minutes*

F00S23 Compute a such that $x^2 + 5x + a = 0$ and $x^2 + ax + 5 = 0$ have exactly one root in common.

F00S24 For what value(s) of a^2 is x real?

$$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = ax$$

Answers:

F00S19	4
F00S20	$(\frac{1}{2}, \frac{1}{2}, -1)$
F00S21	25
F00S22	1
F00S23	-6
F00S24	$a^2 \geq \frac{1}{2}$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER FIVE**

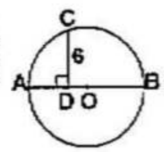
PART I *FALL, 2000* *CONTEST 5* *TIME: 10 MINUTES*

F00S25 Rachel lines up 5 white and 5 black marbles as follows: ○○○○○●●●●●. Each move she makes consists of switching two adjacent marbles at a time. What is the fewest number of moves she needs to change the above into the following arrangement: ●○●○●○●○●○?

F00S26 Define the product function $\prod_{k=1}^{n} k$ as $1\cdot 2\cdot 3\cdot\ldots\cdot n$. Compute the value of $\prod_{k=1}^{100}\frac{4k^2-4k+1}{4k^2-1}$.

PART II *FALL, 2000* *CONTEST 5* *TIME: 10 MINUTES*

F00S27 Points A, B, and C lie on circle O so that $\overline{AB}$ is a diameter, $\overline{CD}\perp\overline{AB}$, AD < BD, and CD = 6. If the numbers AD, CD, and BD − 1 form an arithmetic progression in that order, compute BD.



F00S28 Compute all the roots of $(x+2)^5 = x^5 + 2^5$.

PART III *FALL, 2000* *CONTEST 5* *TIME: 10 MINUTES*

F00S29 An acute angle of a right triangle contains 60° and the hypotenuse is 12. If the bisector of the larger acute angle divides the right triangle into two regions, compute the area of the larger region.

F00S30 If $\frac{a+b-c}{c}=\frac{b+c-a}{a}=\frac{c+a-b}{b}$ and $abc \neq 0$, compute all numerical values for $\frac{(a+b)(b+c)(c+a)}{abc}$.

ANSWERS:

F00S25	15
F00S26	$\frac{1}{201}$
F00S27	9
F00S28	$0, -2, -1\pm i\sqrt{3}$
F00S29	$12\sqrt{3}$
F00S30	$-1, 8$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | CONTEST NUMBER ONE

SOLUTIONS

F00S1 **Answer: 8,–7.** Since $|N| > 1$, $N^2 - N = 56$. Thus, $N = 7$ or -8. Both check.

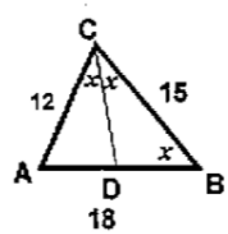
F00S2 **Answer: 16.** If a and r represent the first term and the common ratio respectively, then $ar^2 - ar = 12$, and $ar^4 - ar^3 = 27$. But $ar^4 - ar^3 = r^2(ar^2 - ar) = r^2(12) = 27$. Since the terms increase, $r = 1.5$. By substitution, $a = 16$.

F00S3 **Answer: 120°.** $180(n-2) > 2400 \Rightarrow 3(n-2) > 40 \Rightarrow (n-2) > 13+ \Rightarrow n > 15+$. Therefore the polygon has 16 sides, the sum of the angles is $180(14) = 2520°$ and the angle contains 120°. If $n = 17$, the polygon is not convex. Alternately, $2400 \div 180$ leaves a remainder of 60°. Thus 120° are needed to reach the next multiple of 180°.

F00S4 **Answer: 778.** The required set is the complement of the set of all multiples of 6 or 9. Any multiple of both 6 and 9 is also a multiple of 18 and appears in both sets of multiples. From 1 to 1000 there are $[1000 \div 6] = 166$ multiples of 6, $[1000 \div 9] = 111$ multiples of 9, and $[1000 \div 18] = 55$ multiples of 18. Then, with $166 + 111 - 55 = 222$ multiples of 6 or 9, there are $1000 - 222 = 778$ nonmultiples of 6 or 9 in the set.

F00S5 **Answer: 7.** For B boys and G girls, $B = 5(G-1)$ and $B - 1 = 2G$. Then $5(G-1) - 1 = 2G \Rightarrow G = 2 \Rightarrow B = 5 \Rightarrow G + B = 7$.

F00S6 **Answer: 2x.** Bisect ∠C. By the angle bisector theorem, AD:DB = 4:5. Then $4n + 5n = 18$, so AD = 8 and DB = 10. Notice that 8:12 = 12:18, so AD:AC = AC:AB. Since ∠A ≅ ∠A, ΔACD ~ ΔABC. Thus, ∠ACD ≅ ∠CBD. Because m∠B = x, m∠ACD = x, m∠BCD = x, and m∠ACB = $2x$.

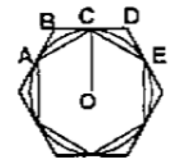


SOLUTIONS

F00S7 Answer: $5^{2x}, 2^{5x}, 4^{3x}, 3^{4x}$. $2^{5x} = (2^5)^x = 32^x$; $3^{4x} = (3^4)^x = 81^x$; $4^{3x} = (4^3)^x = 64^x$; and $5^{2x} = (5^2)^x = 25^x$. Since $x > 1$, $25^x < 32^x < 64^x < 81^x$.

F00S8 Answer: 8. $\sqrt{12+8\sqrt{2}} + \sqrt{12-8\sqrt{2}} = \sqrt{12+2\sqrt{32}} + \sqrt{12-2\sqrt{32}} = \left(\sqrt{8}+\sqrt{4}\right)+\left(\sqrt{8}-\sqrt{4}\right) = 2\sqrt{8}$. Therefore $N = 8$. [***Theorem:* $\sqrt{a}+\sqrt{b} = \sqrt{(a+b)+2\sqrt{ab}}$. To prove it, square both sides and equate like terms.]

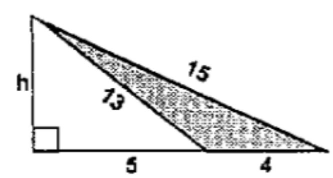
F00S9 Answer: $12\sqrt{3}$. The perimeter of the outer hexagon is 24 so BD = 4. Since triangle OBD (not shown) is equilateral, altitude OC = $2\sqrt{3}$. Then in equilateral triangle OCE (not shown), CE = OC = $2\sqrt{3}$, so the perimeter is $12\sqrt{3}$.



F00S10 Answer: 17. Since $10_6 = 3_6 \cdot 2_6$, and $100_6!$ has fewer factors of 3_6 than of 2_6, the number of terminal zeros is equal to the number of times that 3_6 appears as a factor. Since $100_6! = 36_{10}!$, there are 36 factors, of which 12 are multiples of 3. Of these 12 factors, 4 (every third factor) are also multiples of 9, which yields 4 more factors of 3. Of these 4 factors, 1 (every third factor) is also a multiple of 27, which yields 1 more factor of 3. Thus, $100_6!$ contains 12+4+1 = 17 factors of 3_6 and there are 17 terminal zeros.

F00S11 Answer: 12. This triangle was chosen because it represents the intersection of two overlapping right triangles, the 5-12-13 and the 9-12-15. To derive by technical means: let k = the segment marked 5. Then $h^2 = 13^2 - k^2 = 15^2 - (k+4)^2$, so $k = 5$. Finally, by the Pythagorean Theorem, $h = 12$.

Alternately, by Hero's Formula, the area of the triangle is 24. Then $24 = \frac{1}{2}(4)(h)$, so $h = 12$.



F00S12 Answer: 6, –9.

$$\sqrt{n(n+1)(n+2)(n+3)+1} = 55$$
$$\sqrt{(n^2+3n)(n^2+3n+2)+1} = 55$$
$$\sqrt{(n^2+3n)^2+2(n^2+3n)+1} = 55$$
$$\sqrt{(n^2+3n+1)^2} = 55$$
$$n^2+3n+1 = 55$$
$$n^2+3n-54 = 0$$

Or: Let $n = x - \frac{3}{2}$

$$\sqrt{(x-\tfrac{3}{2})(x-\tfrac{1}{2})(x+\tfrac{1}{2})(x+\tfrac{3}{2})+1} = 55$$
$$\sqrt{(x^2-\tfrac{9}{4})(x^2-\tfrac{1}{4})+1} = 55$$
$$\sqrt{x^4-\tfrac{5}{2}x^2+\tfrac{25}{16}} = 55$$
$$x^2-\tfrac{5}{4} = 55$$
$$x^2 = 56.25, \text{ so } x = \pm 7.5$$

Thus, n = 6 or –9.

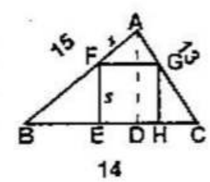
SOLUTIONS

F00S13 **Answer: 37.** $2000 \to 4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$, and so on. Since 4 is both the 2nd and 10th terms, these values repeat in cycles of 8 terms. Then 4 will appear 96 terms after the 2nd term. Since 4 is the 98th term, the 100th term is 37.

F00S14 **Answer: 90.** Multiply the second equation by 2 and subtract equations: Since $2(2a + 4b + c) - (3a + 7b + c) = a + b + c$, the value is $2(205) - (320) = 90$.

F00S15 **Answer: 37.** **METHOD 1:** The maximum possible score is $100 \times 7 = 700$ points. Ashley's score of 330 implies that she lost 370 points from the maximum. Each wrong answer actually cost her 10 points: the 7 she does not score plus the 3 that is deducted. Then she has $370 \div 10 = 37$ questions wrong. **METHOD 2:** For w wrong answers, $7(100 - w) - 3w = 330$, which yields $w = 37$. **METHOD 3:** Getting all right or all wrong yields scores of 700 and -300, an interval of 1000 points. A score of 330 is 370 points less than 700. Then $370{:}1000 = x{:}100$. **METHOD 4:** Bracketing. Getting half of all questions correct produces 200 points: too low. Getting 3/4 correct produces 450 points: too high. Since 330 is about halfway between these numbers, try the average of 50 and 75. This under-and-over method of Bracketing is used by artillery to locate a target when measurements of distances are not available.

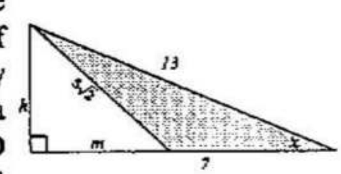
F00S16 **Answer:** $\frac{84}{13}$. The 13-14-15 triangle was chosen because it represents the union of two nonoverlapping right triangles, the 5-12-13 and the 9-12-15. The derivation is similar to that of problem 11 from the last contest. Altitude AD = 12.



Because the $\triangle AFG \sim \triangle ABC$, $AF{:}FG = AB{:}BC$ so that $x{:}s = 15{:}14$. Also, because $\triangle BFE \sim \triangle BAD$, $BF{:}FE = BA{:}AD$ so that $(15 - x){:}s = 15{:}12$. Then $15s = 14x = 12(15 - x)$, so that $x = \frac{90}{13}$. Therefore, $s = \frac{84}{13}$.

F00S17 **Answer: 20.** $n^2 + 6n + 32 = (n + 3)^2 + 23$. Since 23 divides both $n^2 + 6n + 32$ and 23 then it divides $(n + 3)^2$. Therefore it divides $n + 3$. Thus, the least positive integral value of n is 20.

F00S18 **Answer: 17.5** or $\frac{35}{2}$ or $17\frac{1}{2}$. **METHOD I:** If x is the angle formed by the sides of length 7 and 13, then, by the Law of Cosines, $50 = 49 + 169 - 2(7)(13)(\cos x)$. Then $\cos x = \frac{12}{13}$. By the Pythagorean relationship, $\sin x = \frac{5}{13}$. Since the area of a triangle is $\frac{1}{2}ab\sin C$, then $A = \frac{1}{2}(7)(13)(\frac{5}{13}) = 17.5$. **METHOD II:** Since $(5\sqrt{2})^2 + 7^2 > 13^2$, the triangle is obtuse. The side 13 suggests the Pythagorean Triple 5,12,13. Let $h = 5$ and $m = 12 - 7 = 5$. Then the side $5\sqrt{2}$ confirms that these are the correct dimensions. Thus, the area is $\frac{1}{2}(5)(7) = 17.5$.

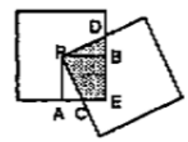


SOLUTIONS

F00S19 **Answer: 4.** Any number divisible by 24 is also divisible by 3. Then 3|(1+7+5+A+A), so 3|(2A + 13). Thus A = 1 or 4 or 7. Also, Any number divisible by 24 is also divisible by 8. Then 8|5AA. Substitute: 8 does not divide 511 or 577, but it does divide 544. Thus, A = 4.

F00S20 **Answer:** $(\frac{1}{2},\frac{1}{2},-1)$. **METHOD I:** Rewrite the given expression: $f(n) - f(n-1) = n$. Then $f(2) - f(1) = 2$, $f(3) - f(2) = 3$, $f(4) - f(3) = 4$, ... ,$f(n) - f(n-1) = n$. Add the equations: $f(n) - f(1) = 2 + 3 + 4 + \ldots + n$, which is an arithmetic progression. Thus, $f(n) - 0 = \frac{n-1}{2}(2+n) = \frac{1}{2}(n^2 + n - 2)$. **METHOD II:** By substitution, the first 5 terms are 0, 2, 5, 9, and 14, which are 1 less than the corresponding triangular numbers. Thus the n^{th} term is $\frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2}$.

F00S21 **Answer: 25.** Drop perpendiculars $\overline{PA}$ and $\overline{PB}$. Since $\overline{PA} \cong \overline{PB}$, $\angle PAC \cong \angle PBD$, and $\angle APC \cong \angle BPD$, $\Delta PAC \cong \Delta PBD$. Then the area of square PAEB = the area of the shaded region = ¼(100) = 25.



F00S22 **Answer: 1.** $\Sigma(\sin K + \cos K) = (\Sigma \sin K) + (\Sigma \cos K)$. Draw the graph of $y = \sin K$ for the interval from 0 to 360, inclusive. Since it has rotational symmetry about (180°,0), $\sin K + \sin(360 - K) = 0$. Then $\Sigma \sin K = 0$. Similarly, draw the graph of $y = \cos K$ for the interval from 0 to 360, inclusive. In the interval $0 < K < 180$, it has rotational symmetry symmetry about (90°,0), and in the interval $180 < K < 360$, it has rotational symmetry about (270°,0). Then $\Sigma \cos K = 0$ for all points between 0 and 360 except if $K = 0$, $K = 180$, and $K = 360$. Then $0 + 1 + {}^{-}1 + 1 = 1$.

F00S23 **Answer:** -6. Since the equations have a root in common, $x^2 + 5x + a = x^2 + ax + 5$. Then $(a - 5)x = a - 5$, which implies that either $a = 5$ or $x = 1$. If $a = 5$, then the equations have both roots in common. However, if $x = 1$, then $1^2 + 5(1) + a = 0$, making $a = -6$. Checking by substitution, the roots of one equation are 1 and 5, and of the other equation are 1, and -6. If $a = -6$, then the equations have exactly one root in common.

F00S24 **Answer:** $a^2 \geq \frac{1}{2}$. $\left(\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}\right)\left(\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\right) = ax$ yields $\frac{a+\sqrt{a^2-x^2}}{x} = ax$. Then $\sqrt{a^2 - x^2} = ax^2 - a$. Squaring and simplifying produces $a^2x^4 = 2a^2x^2 - x^2$. Now $x \neq 0$, since otherwise the original denominator would be zero. Thus, $a^2x^2 = 2a^2 - 1$. By the quadratic formula, $x = \pm\sqrt{\frac{2a^2-1}{a^2}} = \frac{\pm\sqrt{2a^2-1}}{a}$. If $a^2 < \frac{1}{2}$, then x is imaginary. Therefore, $a^2 \geq \frac{1}{2}$

New York City Interscholastic Math League

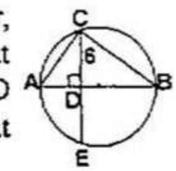
Senior A Division | **Contest Number Five**

Solutions

F00S25 **Answer: 15.** One method is: number the marbles left to right from 1 to 10. To move the #6 marble into the #1 position takes 5 switches. To move the #7 marble into the #3 position takes 4 switches. To move the #8 marble into the #5 position takes 3 switches. To move the #9 marble into the #7 position takes 2 switches. To move the #10 marble into the #9 position takes 1 switch. That's a total of 15 switches. Since each black marble moves directly from initial position to final position, fewer moves are not possible.

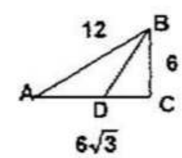
F00S26 **Answer:** $\frac{1}{201}$. $\prod_{k=1}^{100}\frac{4k^2-4k+1}{4k^2-1}=\prod_{k=1}^{100}\frac{(2k-1)(2k-1)}{(2k-1)(2k+1)}=\prod_{k=1}^{100}\frac{2k-1}{2k+1}=\frac{1}{3}\cdot\frac{3}{5}\cdot\frac{5}{7}\cdots\frac{199}{201}=\frac{1}{201}$.

F00S27 **Answer: 9.** Draw $\overline{CA}$ and $\overline{CB}$. Then in right ΔACB, $CD^2 = AD\cdot BD$. [or, extend semichord CD to become chord CE. Then $\overline{AB}$ bisects $\overline{CE}$, so that $AD\cdot BD = CD\cdot DE$, or $CD^2 = AD\cdot BD$.] Either way, $AD\cdot BD = 36$. Also, $6 - AD = (BD - 1) - 6$, which produces $AD + BD = 13$. The two positive numbers that add to 13 and multiply to 36 are 4 and 9. Since $BD > AD$, $BD = 9$.



F00S28 **Answer: $0, -2, -1\pm i\sqrt{3}$.** Expanding, $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 = x^5 + 32$. Then $x^4 + 4x^3 + 8x^2 + 8x = 0$. Thus, one root is 0. Grouping, $x^3 + 2x^2 + 2x^2 + 4x + 4x + 8 = x^2(x+2) + 2x(x+2) + 4(x+2) = (x+2)(x^2 + 2x + 4) = 0$. A second root is -2, and the quadratic formula yields the other two roots.

F00S29 **Answer: $12\sqrt{3}$.** Triangle ABC is a $30° - 60° - 90°$ triangle, so $BC = 6$ and $AC = 6\sqrt{3}$. By the Angle Bisector Theorem, $AD{:}DC = 12{:}6$. Thus $AD = \frac{2}{3}$ of $6\sqrt{3} = 4\sqrt{3}$. The area of triangle ABD is $\frac{1}{2}(4\sqrt{3})(6) = 12\sqrt{3}$.



F00S30 **Answer: $-1, 8$.** Since $\frac{a+b-c}{c}=\frac{b+c-a}{a}$, then $a^2 + ab - ac = bc + c^2 - ac$. This is equivalent to $(a^2-ac) + (ab-bc) + (ac-c^2) = 0$, which produces $(a-c)(a+b+c) = 0$. Thus either $a = c$ or $a + b = -c$. If $a = c$, $\frac{b+c-a}{a}=\frac{c+a-b}{b}$ yields $(b+2c)(b-c) = 0$. Thus, if $a = c$, then either $b = -2c = -2a$ or $b = c = a$. Substituting $b = -2c = -2a$ into $\frac{(a+b)(b+c)(c+a)}{abc}$ produces the value -1, as does substituting $a + b + c = 0$. On the other hand, substituting $b = c = a$ into $\frac{(a+b)(b+c)(c+a)}{abc}$ produces $\frac{(2a)(2b)(2c)}{abc} = 8$.

Note that by adding 1 to each side of $\frac{a+b-c}{c}=\frac{b+c-a}{a}=\frac{c+a-b}{b}$ produces $\frac{a+b}{c}=\frac{b+c}{a}=\frac{c+a}{b}$, which implies that $a = b = c$.