New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2000**

PART I: 10 minutes **NYCIML Contest One** **Fall 2000**

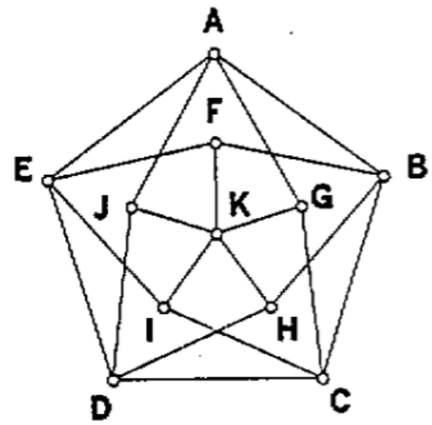
F00J1. Maria leaves home and drives at a constant 30 mph. Two hours after she leaves, her husband John realizes that she forgot her house keys and drives at a constant x mph after leaving home. He catches up to her in 3.5 hours. Compute x.

F00J2. Kellyland uses Yellow Pig Dollars as currency. These come in bills for all odd integer amounts of Yellow Pig Dollars from 1 to 17. Compute the number of different ways there are of giving change to a Yellow Pig 17 Dollar bill, so that a Yellow Pig 5 Dollar bill is included?

PART II: 10 minutes **NYCIML Contest One** **Fall 2000**

F00J3. Compute the number of even positive factors of 2000.

F00J4. Compute the minimum number of colors necessary to color the 11 vertices, labeled A through K, of the graph, so that no two adjacent vertices have the same color? (For vertices to be adjacent, there has to be a line segment connecting them directly. For example, A and G are adjacent, but A and C are not adjacent, and A and F are not adjacent.) This is called the chromatic number of a graph.



PART III: 10 minutes **NYCIML Contest One** **Fall 2000**

F00J5. Compute the remainder when 8^{2000} is divided by 13.

F00J6. The angle bisectors of 30-60-90 right triangle ABC with hypotenuse $BC = 10$ meet in point D. Compute AD.

ANSWERS:

J1. $\frac{330}{7}$ or $47\frac{1}{7}$

J2. 15

J3. 16

J4. 4

J5. 1

J6. $\frac{5\sqrt{6}-5\sqrt{2}}{2}$

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JUNIOR DIVISION | **CONTEST NUMBER TWO** | **FALL 2000**

PART I: 10 minutes | **NYCIML Contest Two** | **Fall 2000**

F00J7. A club meeting has 12 men and 9 women present. The women shake hands with everyone, but no man shakes hands with another man. Compute the number of unique handshakes that are exchanged.

F00J8. A beaker contains 27.5 ounces of an acid-water solution that is 40% acid. Some water evaporates and the new solution is 55% acid. Compute the number of ounces in the 55% solution.

PART II: 10 minutes | **NYCIML Contest Two** | **Fall 2000**

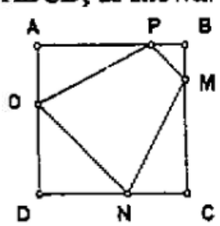
F00J9. Compute all integers x such that $x^{(x-1)^2} = x^{25}$.

F00J10. In rectangle JKLM, JK=65. Point P is on $\overline{LM}$, so that LP:PM=1:4. Compute the area of right triangle JKP.

PART III: 10 minutes | **NYCIML Contest Two** | **Fall 2000**

F00J11. The sides of a triangle are in the ratio 3:5:6. The area of the triangle is $\sqrt{504}$. Compute the length of the largest side of the triangle.

F00J12. Isosceles trapezoid MNOP, PO=MN, with area = 70 is inscribed in square ABCD, as shown. The area of ABCD is 144 and BM=PB=2. Compute CN. (CN>0.)



ANSWERS:

J7. 144
J8. 20
J9. -4,0,1,6

J10. 845
J11. $6\sqrt{3}$.
J12. 4

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JUNIOR DIVISION — **CONTEST NUMBER THREE** — **FALL 2000**

PART I: 10 minutes — **NYCIML Contest Three** — **Fall 2000**

F00J13. Compute the number of rectangles in the following strip.

F00J14. The planet Melmac has a year with but 8 days. If three residents of Melmac are in a room, compute the probability that at least two were born on the same day of the year.

PART II: 10 minutes — **NYCIML Contest Three** — **Fall 2000**

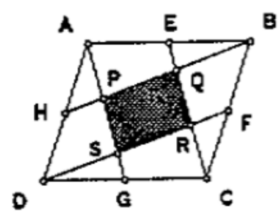
F00J15. Compute all prime numbers greater than 200 and less than 218.

F00J16. Right triangle ABC has hypotenuse $\overline{BC}$ of length 100, and altitude $\overline{AD}$ of length 40. Compute the length of the shorter leg of the triangle.

PART III: 10 minutes — **NYCIML Contest Three** — **Fall 2000**

F00J17. On the planet Ork, all numbers are in base 8. On the planet Dork, all numbers are in base x. If Quark has 247 bars of gold pressed latinum on Ork, which would be 1132 bars of gold pressed latinum on Dork, compute x.

F00J18. The area of parallelogram ABCD is 70. E, F, G, and H are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Compute the area of parallelogram PQRS, as shown in the diagram.



ANSWERS:

J13. 55

J14. $\frac{11}{32}$

J15. 211

J16. $20\sqrt{5}$

J17. 5

J18. 14

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JUNIOR DIVISION | **CONTEST NUMBER ONE SOLUTIONS** | **FALL 2000**

F00J1. $30 \cdot 5.5 = x \cdot 3.5 \Rightarrow x = \frac{330}{7}$ or $47\frac{1}{7}$.

F00J2. We must compute the number of ways to express 12 as a sum of positive odd numbers. These are 11+1 = 9+3 = 9+3x1 = 7+5 = 7+3+2x1 = 7+5x1 = 2x5+2x1 = 5+2x3+1 = 5+3+4x1 = 5+7x1 = 4x3 = 3x3+3x1 = 2x3+6x1 = 3+9x1 = 12x1. There are 15 ways.

F00J3. $2000 = 2 \cdot 10^3 = 2^4 \cdot 5^3$. There are thus (4+1)(3+1)=20 positive integral factors. Of these, only 1,5,25, and 125 are odd, which leaves 16 even factors.

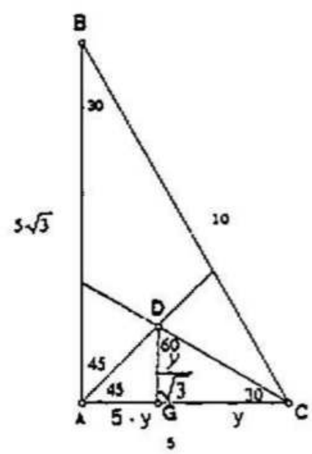
OR

$2000 = 1000 \cdot 2$. The number of positive integral factors of 1000 is 16. Doubling each of these would give an even factor of 2000. Thus there are 16 even factors.

F00J4. The 5 exterior vertices require at least 3 colors, and there is only one general method of coloring them with just 3 colors. Attempting to extend this coloring to the rest of the graph requires all three colors to appear among the interior 5 vertices, forcing a fourth color for the middle vertex. Thus 4 colors is the minimum.

F00J5. $8^2 = 64 \equiv -1 \pmod{13}$

$8^{2000} = (8^2)^{1000} \equiv (-1)^{1000} \equiv 1 \pmod{13}$ so the remainder is 1.



F00J6. Let GC = y, AG = $5 - y$. AG = DG, so

$$\frac{y\sqrt{3}}{3} = 5 - y \rightarrow y = \frac{15}{3+\sqrt{3}} \rightarrow y = \frac{15-5\sqrt{3}}{2}$$

$$\rightarrow 5 - y = \frac{5\sqrt{3}-5}{2}$$

To find AD, we multiply by $\sqrt{2}$ and get $\frac{5\sqrt{6}-5\sqrt{2}}{2}$.

OR

The angle bisectors will meet at the center of the inscribed circle. Let this circle be tangent to $\overline{AB}$ and $\overline{AC}$ at S and T, respectively. $ASDT$ is a square with a side equal to the radius of the inscribed circle (the inradius) and with AD as the diagonal. In a right triangle, the inradius is half the difference between the sum of the legs and the hypotenuse. $BC = 10$, so the lengths of the legs are 5 and $5\sqrt{3}$, and

$$AD = \sqrt{2}\,\frac{5+5\sqrt{3}-10}{2} = \frac{5\sqrt{6}-5\sqrt{2}}{2}.$$

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JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2000**

SOLUTIONS

F00J7. The number of handshakes among the women will be ${}_9C_2 = 36$. Each of the 9 women will also shake hands with 12 men, an additional 108 handshakes. There will be **144** handshakes exchanged.

F00J8. There are 11 ounces of acid in the solution. The new solution will be 45% water. Let x = the amount of water in the new solution. $.45(x+11) = x$. Thus $x = 9$. There are **20** ounces in the 55% acid solution.

F00J9. This is clearly true if $x=0$ or 1. It is also true if

$(x-1)^2 = 25 \Rightarrow x-1 = \pm 5 \Rightarrow x = 6$ or $x = -4$. $\{-4,0,1,6\}$

F00J10. LP+PM = JK = 65, so LP=13 and PM= 52. Triangles JMP and PLK are similar, and calling JM = LK= x, we obtain x:52=13:x, so x=26. The area of the triangle is

$\frac{1}{2} \cdot 65 \cdot 26 = 845$.

F00J11. Let $3x$, $5x$, and $6x$ be the lengths of the sides. By Heron's Formula,

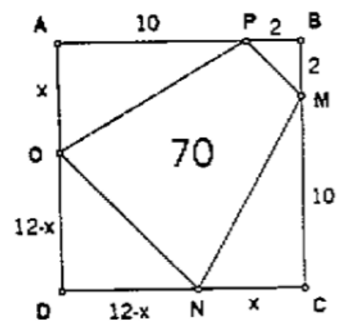
$\sqrt{7x \cdot 4x \cdot 2x \cdot x} = \sqrt{504} \Rightarrow 2x^2\sqrt{14} = \sqrt{504} \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$

Thus the largest side is $\mathbf{6\sqrt{3}}$.

F00J12.

Let x = NC, 12-x = ND. $144 = 70 + 10x + 2 + \frac{1}{2}(12-x)^2$

$\frac{1}{2}x^2 - 2x + 144 = 144 \rightarrow x^2 - 4x = 0 \rightarrow x = 4$



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JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2000**

SOLUTIONS

F00J13. Let the length of the smallest rectangle be one unit. There are 10 rectangles with length 1, 9 with length 2, 8 with length 3, etc. $10+9+8+...+1=\mathbf{55}$.

F00J14 Each resident can have one of eight birthdays, giving $8\cdot 8\cdot 8 = 512$ possibilities. There are ${}_8C_3 = 56$ possible ways of choosing three different birthdays, giving $6\cdot 56 = 336$ possible ways for three people to have these (different) birthdays. All $512-336=176$ other birthday sets will have at least one shared birthday. Thus the probability is $\frac{176}{512}=\mathbf{\frac{11}{32}}$.

OR

The probability that resident two does not have the same birthday as resident one is $\frac{7}{8}$. The probability that resident three does not have the same birthday as resident one or resident two is $\frac{6}{8}$. Therefore, the probability that no two have the same birthday is $\frac{6}{8}\cdot\frac{7}{8}=\frac{21}{32}$. Therefore, the probability that at least two have the same birthday is $1-\frac{21}{32}=\mathbf{\frac{11}{32}}$.

F00J15. We need only look at odd numbers that do not end in 5, in the range. Thus we check 201, 203, 207, 209, 211, 213 and 217. By the divisibility rule for 3, we eliminate 201, 207, and 213. By the divisibility rule for 11, we eliminate 209. 203 is divisible by 7, as is 217, so these are eliminated. Only **211** remains.

F00J16. Let x and y be the lengths of the legs, $x>y$. $x^2+y^2=10000$,
$\frac{1}{2}xy=\frac{1}{2}\cdot 40\cdot 100=2000$, so $2xy=8000$, giving $x^2+2xy+y^2=18000$, so
$x+y=60\sqrt{5}$. Also, $x^2-2xy+y^2=2000$, so $x-y=20\sqrt{5}$.
$(x+y)-(x-y)=2y=40\sqrt{5}$, so $\mathbf{y=20\sqrt{5}}$

F00J17. $1132_x=247_8=167_{10}$
Clearly, $x<8$. Since 3 is a digit base x, $x>3$. Checking all four possibilities, we find $1132_5=125+25+15+2=167_{10}$, so $\mathbf{x=5}$.

F00J18. Extend the lines as shown in the figure. $EQ=EE'$, $FR=FF'$, $GS=GG'$, and $PH=HH'$. $\Delta EQB\cong\Delta EE'A$, $\Delta FRC\cong\Delta FF'B$, $\Delta GSD\cong\Delta GG'C$, and $\Delta HPA\cong\Delta HH'D$ (by s.a.s.) Therefore Area(ABCD)=Area(AE'QBF'RCG'SDH'P). Since AE'QBF'RCG'SDH'P is composed of five congruent parallelograms, one of which is PQRS, Area(PQRS) = $\frac{1}{5}$Area(ABCD) = **14**.