NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER ONE**

PART I *SPRING, 1999* *CONTEST 1* *TIME: 10 MINUTES*

S99S1 In the addition of four digit numbers at the right, different letters represent different digits. What is the four digit number CCBA?.

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  AABB
+ BACC
  CCBA
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S99S2 Compute the real value of x for which $x^3 - 16x - 585 = 0$.

PART II *SPRING, 1999* *CONTEST 1* *TIME: 10 MINUTES*

S99S3 In an arithmetic progression of 49 terms, the first term is 5 and the common difference is 4. A square is subdivided into 49 congruent squares and each term is written in one of the small squares so that no square is empty. If the integers in every row have the same sum, what is this sum?

S99S4 The roots of $ax^2 + bx + c = 0$ are the reciprocals of the roots of $19x^2 + 41x + 23 = 0$. If a, b, and c are relatively prime positive integers, what is the ordered triple (a, b, c)?

PART III *SPRING, 1999* *CONTEST 1* *TIME: 10 MINUTES*

S99S5 **S** is the set of all integers which are both perfect squares and perfect cubes of integers. Compute the fifth smallest member of **S**.

S99S6 The diagonal of an 8×15 rectangle cuts it into two triangles. A circle is inscribed in each triangle. Compute the distance between the centers of the two circles.

ANSWERS:

S99S1	9954
S99S2	9
S99S3	707
S99S4	(23, 41, 19)
S99S5	15,625
S99S6	$\sqrt{85}$

New York City Interscholastic Math League

Senior A Division | **Contest Number Two**

Part I *Spring, 1999* *Contest 2* *Time: 10 Minutes*

S99S7 The radius of a circle is 6. Two other circles, each concentric with and within the original circle, are drawn to form three regions of equal area. Compute the radius of the smallest of the three circles.

S99S8 Given the recursive function $f_{n+1} = (2 + f_n^{-1})^{-1}$, where x^{-1} indicates the reciprocal of x, for all values of x. If $f_1 = 1$, compute the value of f_{1000}.

Part II *Spring, 1999* *Contest 2* *Time: 10 Minutes*

S99S9 A certain fraction can be written either as $.\overline{13}$ in base a or as $.\overline{8}$ in base a^2, where $a > 0$ and $.\overline{mn}$ represents the repeating decimal $.mnmnmn\ldots$. Compute this fraction in base 10.

S99S10 In ΔABC, AB = 6, AC = 9, and BC = 10. $\overline{BC}$ contains points D and E so that $\overline{AD}$ is an altitude and $\overline{AE}$ bisects $\angle BAC$. Compute DE.

Part III *Spring, 1999* *Contest 2* *Time: 10 Minutes*

S99S11 Given:

$$\frac{x+1}{x-1} + \frac{y+2}{y-2} = 5$$

$$\frac{y+2}{y-2} + \frac{z+3}{z-3} = 7$$

$$\frac{z+3}{z-3} + \frac{x+1}{x-1} = 6.$$ Compute (x, y, z).

S99S12 In radian measure, compute all values of θ in the interval $0 \le \theta \le 2\pi$ for which $\sin 2\theta + \cos 2\theta$ is maximized.

Answers:

S99S7	$2\sqrt{3}$
S99S8	$1/1999$
S99S9	$1/3$
S99S10	1.25 *or* $1\frac{1}{4}$ *or* $5/4$
S99S11	(3,4,5)
S99S12	$\pi/8$, $9\pi/8$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION **CONTEST NUMBER THREE**

PART I *SPRING, 1999* *CONTEST 3* *TIME: 10 MINUTES*

S99S13 In what base b is the square root of 232_b equal to 14_b?

S99S14 An arithmetic progression of integers contains $2n$ terms. If the sum of the first n terms is subtracted from the sum of the next n terms, the difference is 175. If $1 < n < 175$, compute n.

PART II *SPRING, 1999* *CONTEST 3* *TIME: 10 MINUTES*

S99S15 If the integer N is divided by 2, 3, 4, 5, or 6, the remainder in each case is 1. However if N is divided by 11, the remainder is 0. What is the least positive value of N?

S99S16 Compute all ordered pairs of real numbers (x,y) such that:

$$\left(x+\tfrac{1}{x}\right)\left(y+\tfrac{1}{y}\right)=5$$
$$\left(x+\tfrac{1}{y}\right)\left(y+\tfrac{1}{x}\right)=4.5$$

PART III *SPRING, 1999* *CONTEST 3* *TIME: 10 MINUTES*

S99S17 Tanya tosses 4 fair coins. At least one coin shows heads. In decimal form, find the probability that exactly two coins show tails.

S99S18 Trapezoid ABCD is inscribed in a circle so that base AB is a diameter. The ratio of base DC to DA is 2:1. Compute the ratio CD:AB.

ANSWERS:		
	S99S13	7
	S99S14	5
	S99S15	121
	S99S16	(1,2), (-1,-2), (2,1), (-2,-1), (½,1), (-½,-1), (1,½), (-1,-½)
	S99S17	0.4
	S99S18	$(\sqrt{3}-1):1$ *or* $\sqrt{3}-1$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER FOUR**

PART I | *SPRING, 1999* | *CONTEST 4* | *TIME: 10 MINUTES*

S99S19 Jennifer writes a check for A dollars and B cents, but the cashier reads it as B dollars and A cents. The incorrect amount exceeds the correct amount by S22.77. Compute the largest possible correct amount of the check in dollars and cents.

S99S20 The ellipse $2x^2 + 3y^2 = 6$ and the line $y = mx - 2$ are tangent to each other. If $m > 0$, compute m.

PART II | *SPRING, 1999* | *CONTEST 4* | *TIME: 10 MINUTES*

S99S21 The sum of five different positive odd integers is 105. If the difference between the greatest and least integers is a maximum, find the largest of these integers.

S99S22 In rectangle ABCD, AB = 20 and BC = 24. E is on $\overline{DC}$ so that DE = 18. $\overline{BF}$ is drawn perpendicular to $\overline{AE}$, intersecting $\overline{AD}$ at point F and $\overline{AE}$ at point G. Compute FG.

PART III | *SPRING, 1999* | *CONTEST 4* | *TIME: 10 MINUTES*

S99S23 If $2 \log (x - 3y) = \log 2x + \log 2y$, compute the ratio $x{:}y$.

S99S24 Express $x^4 + x^2y^2 + y^4$ as the product of two polynomials with integral coefficients. Neither factor may be 1.

ANSWERS:

ANSWERS:	S99S19	\$76.99 *or* 76.99
	S99S20	$\sqrt{6}/3$
	S99S21	89
	S99S22	9
	S99S23	9:1 *or* 9
	S99S24	$(x^2 + xy + y^2)(x^2 - xy + y^2)$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER FIVE**

PART I *SPRING, 1999* *CONTEST 5* *TIME: 10 MINUTES*

S99S25 If $\log N = \sum_{i=1}^{1999} \log(2i-1) - \sum_{i=1}^{1999} \log(2i+1)$, compute N.

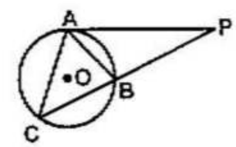
S99S26 Express in terms of n, in simplest terms:

$$(n+2)^4 - 8(n+2)^3 + 24(n+2)^2 - 32(n+2) + 16$$

PART II *SPRING, 1999* *CONTEST 5* *TIME: 10 MINUTES*

S99S27 If 1999 is divided by the positive integer N, the remainder is 19. How many different values can N have?

S99S28 In circle O, PBC is a secant. If chord AB exceeds tangent PA by 2, chord BC by 3, and segment PB by 4, compute AC.



PART III *SPRING, 1999* *CONTEST 5* *TIME: 10 MINUTES*

S99S29 If $\frac{{}_nP_r}{{}_nC_{n-r}} = \frac{r!}{x}$, where $n \geq r > 0$, compute x.

S99S30 Find all ordered triples (a,b,c), where a, b, and c are integers, if:

$$b^c = a^{2c}, \quad 9^b = 9 \cdot 27^c, \quad \text{and} \quad a+b+c = 30.$$

ANSWERS:

S99S25	$^1/_{3999}$
S99S26	n^4
S99S27	24
S99S28	12
S99S29	1
S99S30	(4,16,10), (29,1,0) [*both triples required*]

SOLUTIONS

S99S1 ***Answer:* 9954** From the units column, $C \neq 0$. From the tens column, therefore, $C = 9$. From the hundreds column, $A + A = 9$ implies that 1 was carried and $A = 4$. From either the thousands column or the units column, $B = 5$. Then CCBA = 9954.

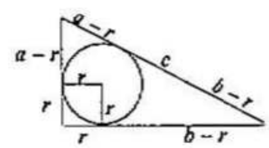
S99S2 ***Answer:* 9** Rewrite as $(x-4)(x)(x+4) = 585$. Therefore, 585 is the product of 3 integers with a common difference of 4. Then $585 = 5\cdot117 = 5\cdot(9\cdot13)$, so $x = 9$.

S99S3 ***Answer:* 707** $a_1 = 5$ and $a_{49} = 5 + 48\cdot4 = 197$. The "average term" is $\frac{1}{2}(5+197) = 101$. The sum of the entries along any one row is $7\cdot101 = 707$. Alternately, the sum of all 49 terms is $\frac{1}{2}(49)(5+197) = 4949$, and the sum of each of the 7 rows is $4949 \div 7 = 707$.

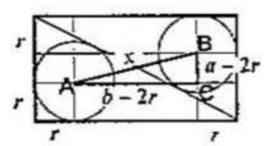
S99S4 ***Answer:* (23, 41, 19)** Suppose r and s are the roots of the original equation. Then the roots of this equation have a sum of $r + s$ and a product of rs, and the form of this equation is $x^2 - (r+s)x + rs = 0$. The final equation has roots $^1/_r$ and $^1/_s$, whose sum is $^{(r+s)}/_{rs}$ and whose product is $^1/_{rs}$. Hence, the new equation in simplest terms is $(rs)x^2 - (r+s)x + 1 = 0$. Thus the equation whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$. In this case the equation is $23x^2 + 41x + 19 = 0$.

S99S5 ***Answer:* 15,625** Any number which is both a square and a cube of integers is actually the sixth power of some integer. Ergo, we require the fifth smallest positive sixth power — namely, $5^6 = 15{,}625$.

S99S6 ***Answer:* $85^{1/2}$** Let a circle be inscribed in a right triangle whose legs are a and b, and whose hypotenuse is c. Using the first diagram, we can prove that the radius $r = \frac{1}{2}(a+b-c)$. In this case $r = \frac{1}{2}(8+15-17) = 3$.



Using the second diagram, we can prove that for ΔABC, $(a-2r)^2 + (b-2r)^2 = x^2$. In this case, $x^2 = (8-2\cdot3)^2 + (15-2\cdot3)^2 = 85$.



Note: Substituting the first formula into the second produces $x^2 = (c-a)^2 + (c-b)^2$.

SOLUTIONS

S99S7 ***Answer:* $2\sqrt{3}$** If r = the radius of the smallest circle, then $3\pi r^2 = \pi 6^2$. Therefore, $r = \sqrt{12} = 2\sqrt{3}$.

S99S8 ***Answer:* 1/1999** $2 + f_n^{-1} = (2 \cdot f_n + 1)/f_n$, so $(2 + f_n^{-1})^{-1} = f_n/(2 \cdot f_n + 1)$. By substitution, $f_1 = 1$, $f_2 = {}^1/_3$, $f_3 = {}^1/_5$, and $f_4 = {}^1/_7$. Thus, $f_k = {}^1/_{(2k-1)}$*. Then $f_{1000} = {}^1/_{(2 \cdot 1000 - 1)} = {}^1/_{1999}$.

*Proof by Mathematical Induction: n=1 is proven above. If $f_k = 1/(2k-1)$, show that $f_{k+1} = 1/(2k+1)$. $f_{k+1} = f_k/(2 \cdot f_k + 1) = [(2k-1)^{-1}] / [2 \cdot (2k-1)^{-1} + 1] = 1/(2k+1)$.

S99S9 ***Answer:* 1/3**

$$S_a = \frac{1}{a} + \frac{3}{a^2} + \frac{1}{a^3} + \frac{3}{a^4} + \frac{1}{a^5} + \frac{3}{a^6} + \cdots$$

$$= \frac{a+3}{a^2} + \frac{a+3}{a^4} + \frac{a+3}{a^6} + \cdots = \frac{a+3}{a^2-1}.$$

$$S_{a^2} = \frac{8}{a^2} + \frac{8}{a^4} + \frac{8}{a^6} + \cdots = \frac{8}{a^2-1}$$

$$\therefore \frac{a+3}{a^2-1} = \frac{8}{a^2-1} \text{ and } a = 5.$$

$$\therefore \frac{8}{5^2} + \frac{8}{5^4} + \frac{8}{5^6} + \cdots = \frac{8}{5^2-1} = \frac{1}{3}.$$

S99S10 ***Answer:* 1.25 or 1¼** By the Angle Bisector Theorem, AB:AC = BE:EC. Thus, BE + EC = $2x + 3x = 10$, so BE = 4. Then, let y = BD so that DC = $10 - y$. By the Pythagorean Theorem, $AD^2 = 6^2 - y^2 = 9^2 - (10-y)^2$, which yields BD = 2.75. Finally, DE = 4 − 2.75 = 1.25.

S99S11 ***Answer:* (3,4,5)** Let $A = \frac{x+1}{x-1}$, $B = \frac{y+2}{y-2}$, and $C = \frac{z+3}{z-3}$ so that A+B = 5, B+C = 7, and C+A = 6. Add all 3 fractions: $2A + 2B + 2C = 18 \Rightarrow A + B + C = 9$. Subtract each of the three equations from A + B + C = 9 to get C = 4, A = 2, and B = 3, respectively. Then $2 = \frac{x+1}{x-1}$, $3 = \frac{y+2}{y-2}$, and $4 = \frac{z+3}{z-3}$ yields $x = 3$, $y = 4$, and $z = 5$.

S99S12 ***Answer:* π/8, 9π/8** If sin 2θ + cos 2θ is maximized, then $(\sin 2\theta + \cos 2\theta)^2$ is also maximized. Expanding, 1 + sin 4θ is maximized. This occurs when sin 4θ = 1. $\therefore \theta = \frac{1}{2}\pi \div 4 = {}^{\pi}/_8$. Sin 2θ + cos 2θ has a period of π, so the other solution in the required interval is ${}^{\pi}/_8 + \pi = {}^{9\pi}/_8$.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER THREE**

SOLUTIONS

S99S13 ***Answer: 7*** $(14_b)^2 = 232_b \Rightarrow (b+4)^2 = b^2+8b+16 \Rightarrow 2b^2+3b+2 = b^2+8b+16$ $\Rightarrow b^2-5b-14=0$. Thus b = 7 or −2. Reject −2 because a base cannot be negative.

S99S14 ***Answer: 5*** Each of the two sets has n elements. If the common difference is d, then each element in the set of second terms exceeds the corresponding element of the set of first terms by nd. This occurs n times. Thus, the sum of the differences between the sets is $n^2d = 175 = 5^2 \cdot 7$ ($1^2 \cdot 175$ contradicts the condition that $1 < n < 175$). Then $n = 5$ and $d = 7$ regardless of the value of the first term.

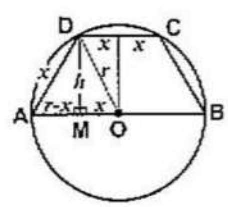
S99S15 ***Answer: 121*** The least common multiple of 2, 3, 4, 5, and 6 is 60, so the least integers that leave remainders of 1 are of the form $60n+1$: 1, 61, 121, 181, 241, $\cdots$. The least of these integers that is divisible by 11 is 121.

S99S16 ***Answer: (1,2), (-1,-2), (2,1), (-2,-1), (½,1), (-½,-1), (1,½), (-1,-½)*** Multiply each equation through by xy and remove parentheses. This yields $x^2y^2+x^2+y^2+1 = 5xy$ and $x^2y^2+2xy+1 = 4.5xy$. Subtract equations: $x^2+y^2-2xy = 0.5xy \Rightarrow 2x^2-5xy+2y^2=0$. $\therefore y = 2x$ or $y = \frac{1}{2}x$. If $y=2x$, then the first equation becomes $(x^2+1)(4x^2+1) = 10x^2$, which produces $x = \pm 1$ or $x = \pm\frac{1}{2}$. If $y = \frac{1}{2}x$, then the first equation becomes $(x^2+1)(x^2+4) = 10x^2$, which produces $x = \pm 1$ or $x = \pm 2$. Substitute $x = \pm 1, \pm 2$, and $\pm\frac{1}{2}$ into the first equation to produce the eight number pairs.

Note: Since x and y are interchangeable in the original equations, the solutions (1,2), (-1,-2), (½,1), and (−½, −1) automatically produce the other four number pairs.

S99S17 ***Answer: 0.4*** Four coins yield 16 cases, of which one, TTTT, is ruled out. There are ${}_4C_2 = 6$ cases for two heads and two tails. Then P(2H, 2T) = 6/15 = 0.4.

S99S18 ***Answer:* $(\sqrt{3}-1):1$ or $\sqrt{3}-1$** We are looking for the ratio $2x:2r = x:r$. Parallel chords intercept congruent arcs, so the trapezoid is isosceles. In $\triangle OMD$, $h^2 = r^2-x^2$ and in $\triangle AMD$, $h^2 = x^2-(r-x)^2$. Equating and simplifying produces $x^2+2xr-2r^2=0$. Rewrite as $(x/r)^2+2(x/r)-2=0$. $\therefore x/r = x:r = (\sqrt{3}-1):1$.



NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER FOUR**

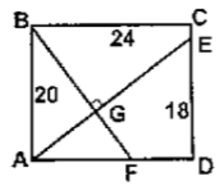
SOLUTIONS

S99S19 ***Answer:* $76.99** If the check is for A dollars and B cents, then: $(100B + A) - (100A + B) = 2277$. Simplifying, $B - A = 23$. Thus the dollar amount exceeds the cent amount by 23 cents. Since the largest possible cent amount is 99¢, the dollar amount must be 76.

S99S20 ***Answer:* $\frac{1}{3}\sqrt{6}$** Substitute: $2x^2 + 3(mx - 2)^2 = 6 \Rightarrow (3m^2 + 2)x^2 - (12m)x + 6 = 0$. Since the line is tangent to the ellipse, the Discriminant $b^2 - 4ac = 0$. $\therefore (12m)^2 - 4(3m^2 + 2)(6) = 0 \Rightarrow m = \frac{1}{3}\sqrt{6}$ since $m > 0$.

S99S21 ***Answer:* 89** To maximize the difference, minimize the smallest integer, 1, and maximize the greatest integer. To maximize the greatest integer, minimize the other three odd integers. Hence, $105 - (1 + 3 + 5 + 7) = 89$.

S99S22 ***Answer:* 9** In ΔADE, $AD = 24$ and $DE = 18$ imply $AE = 30$ (Note the 3-4-5 Pythagorean Triple). $\Delta BFA \sim \Delta AED$, so $BF:BA = AE:AD$. Then $BF:20 = 30:24$, so $BF = 25$. Also $\Delta BGA \sim \Delta ADE$, so $BG:BA = AD:AE$. Then $BG:20 = 24:30$, so $BG = 16$. Finally, $FG = BF - BG = 9$.



S99S23 ***Answer:* 9** $2 \log (x - 3y) = \log 2x + \log 2y \Rightarrow \log (x - 3y)^2 = \log (4xy) \Rightarrow x^2 - 6xy + 9y^2 = 4xy \Rightarrow x^2 - 10xy + 9y^2 = 0 \Rightarrow x = 9y$ or $x = y$. Reject $x = y$ because that would cause $x - 3y$ to be negative. Thus $x = 9y \Rightarrow x/y = x:y = 9$.

S99S24 ***Answer:* $(x^2 + xy + y^2)(x^2 - xy + y^2)$** $x^4 + x^2y^2 + y^4 = (x^4 + 2x^2y^2 + y^4) - x^2y^2 = (x^2 + y^2)^2 - (xy)^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy)$.

SOLUTIONS

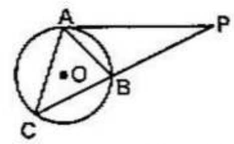
S99S25 ***Answer:* 1/3999** $\sum_{i=1}^{1999}\log(2i-1)-\sum_{i=1}^{1999}\log(2i+1) = (\log 1+\log 3+\log 5+\cdots+\log 3997)$
$-(\log 3+\log 5+\log 7+\cdots+\log 3999) = \log 1 - \log 3999 = \log(1/3999)$.

S99S26 ***Answer:* n^4** $(n+2)^4 - 8(n+2)^3 + 24(n+2)^2 - 32(n+2) + 16$
$= (n+2)^4 - 4\cdot 2(n+2)^3 + 6\cdot 2^2(n+2)^2 - 4\cdot 2^3(n+2) + 2^4$
$= [(n+2)-2]^4$
$= n^4$.

S99S27 ***Answer:* 24** The question is tantamount to asking , "How many positive factors of 1980 are greater than 19?" $1980 = 2^2\cdot 3^2\cdot 5\cdot 11$, so 1980 has $3\times 3\times 2\times 2 = 36$ factors*. Twelve of them are 19 or less: 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, and 18, so N can have $36-12=24$ values.

**Note*: Use the theorem: For any positive integer $N = p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdots p_n^{a_n}$, where all p_i are prime numbers, then N has $(a_1+1)(a_2+1)(a_3+1)\cdots(a_n+1)$ positive factors.

S99S28 ***Answer:* 12** If $AB = x$, then $PA = x-2$, $BC = x-3$, and $PB = x-4$. By the Tangent - Secant Proportion, $PA^2 = PB\cdot PC$, so $(x-2)^2 = (x-4)(2x-7) \Rightarrow x^2 - 11x + 24 = 0 \Rightarrow x = 8$ or $x = 3$. Reject 3 since $PB > 0$, so that $PA = 6$, $PB = 4$, $PC = 9$ and $BA = 8$. Next $\Delta PBA \sim \Delta PAC$, so $PA:PC = PB:PA = BA:AC$. $\therefore 6:9 = 4:6 = 8:AC \Rightarrow AC = 12$.



S99S29 ***Answer:* 1** ${}_nP_r\cdot x = {}_nC_{n-r}\cdot r! \Rightarrow {}_nP_r\cdot x = {}_nC_r\cdot r! \Rightarrow {}_nP_r\cdot x = ({}_nP_r/r!)\cdot r! \Rightarrow {}_nP_r\cdot x = {}_nP_r \Rightarrow x = 1$.

S99S30 ***Answer:* (4,16,10) AND (29,1,0)** (1) $b^c = a^{2c}$, so $b = a^2$ if $c \neq 0$.
(2) $9^b = 9\cdot 27^c \Rightarrow 3^{2b} = 3^2\cdot 3^{3c} \Rightarrow 2b = 2 + 3c \Rightarrow c = \frac{1}{3}(2b-2) \Rightarrow c = \frac{1}{3}(2a^2-2)$.
(3) By substitution, $a + b + c = 30 \Rightarrow a + a^2 + \frac{1}{3}(2a^2-2) = 30 \Rightarrow 5a^2 + 3a - 92 = 0$ $\Rightarrow a = 4$ or $^-4.6$. Reject $^-4.6$ as nonintegral. Thus, $(a, b, c) = (4, 16, 10)$. That is true if $c \neq 0$. However, if $c = 0$, then $b = 1$ and $a = 29$. Thus both triples are needed.

May 3, 1999

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1999 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Senior A	S99S5	4096

Have a great summer!
MATH IS # 1

Sincerely yours,

Richard Geller

Secretary, NYCIML