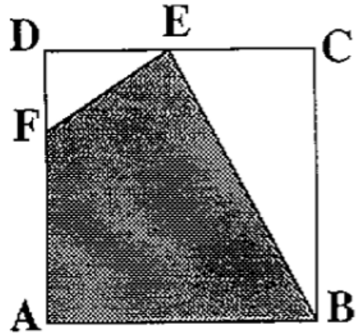
New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 1999**

PART I: 10 Minutes **NYCIML Contest One** **Spring 1999**

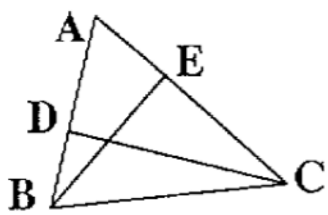
S99J1. In the game of NIM14 there are fourteen cards in a pile. Two players alternate turns taking 1,2,3,4, or 5 cards from the pile each time. The winner is the one who takes the last card. If you go first, how many cards should you take to guarantee that you win?



S99J2. Point E is selected on side $\overline{DC}$ of square ABCD shown on the right so that DE = 5 and EC = 6. Point F is selected on side $\overline{DA}$ so that DF = 2. Find the shaded area.

PART II: 10 Minutes **NYCIML Contest One** **Spring 1999**

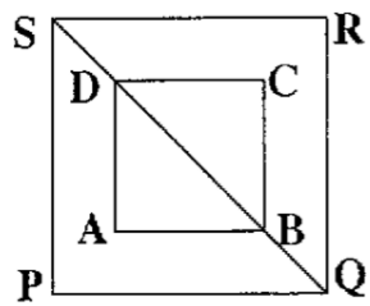
S99J3. When written as a fraction in reduced form, $\frac{5+10+15+...+200}{7+14+21+...+280}$ is $\frac{a}{b}$. Compute the value of a + b.



S99J4. In the diagram on the right, $\overline{BE} \perp \overline{AC}$ and $\overline{CD} \perp \overline{AB}$. If AB = 8, AD = 5, and AE = 4, compute the length of $\overline{EC}$.

PART III: 10 Minutes **NYCIML Contest One** **Spring 1999**

S99J5. Square ABCD is constructed by choosing two points B and D such that $DB = \frac{1}{2}SQ$ and B and D are on a diagonal $\overline{SQ}$ of square PQRS, as shown on the right. If a point is selected at random inside of square PQRS, compute the probability that the point is not inside square ABCD.



S99J6. The fraction $\frac{(2^3-1)(3^3-1)(4^3-1)...(20^3-1)}{(2^3+1)(3^3+1)(4^3+1)...(20^3+1)}$ can be rewritten as $\frac{p}{q}$, where p and q are both positive and have no common factors other than 1. Compute the value of $\frac{p}{q}$.

Answers

1. 2	3. 12	5. $\frac{3}{4}$
2. 83	4. 6	6. $\frac{421}{630}$

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JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 1999**

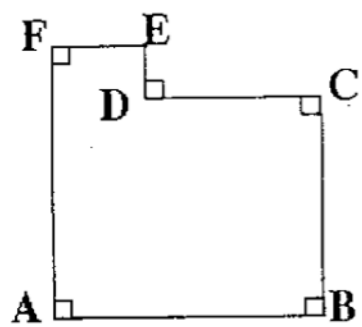
PART I: 10 Minutes **NYCIML Contest Two** **Spring 1999**

S99J7. In the game of NIM 200 there are two hundred cards in a pile. Two players alternate turns taking 1,2,3,4, or 5 cards from the pile each time. The winner is the one who takes the last card. If you go first, how many cards should you take to guarantee that you win?

S99J8. A farmer paid $7400 for a sheep, a cow and an ox. The cow was worth twelve sheep, and the ox was worth two cows. A week later he bought ten sheep, three cows and two oxen. How much did he pay if the prices per animal did not change?

PART II: 10 Minutes **NYCIML Contest Two** **Spring 1999**

S99J9. The figure in the diagram on the right, was cut from a square whose side has length 12. Compute the perimeter of ABCDEF.

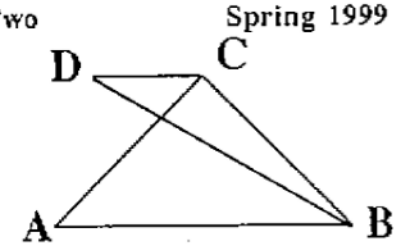


S99J10. The number $3^{10}+3^{8}-10$ can be rewritten as $p^{a}q^{b}r^{c}$, where p, q, and r are prime numbers. Compute the value of $p+q+r+a+b+c$.

PART III: 10 Minutes **NYCIML Contest Two** **Spring 1999**

S99J11. If $\sqrt{12+\sqrt{x}}=5$, compute the value of x.

S99J12. In the diagram on the right, $\overline{AC} \perp \overline{CB}$. $\overline{AC} \cong \overline{BC}$, $\overline{AB} \cong \overline{BD}$, and $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$. Compute $m\angle BDC$.



Answers		
7. 2	9. 48	11. 169
8. $18,800	10. 57	12. 30

New York City Interscholastic Mathematics League

JUNIOR DIVISION CONTEST NUMBER THREE SPRING 1999

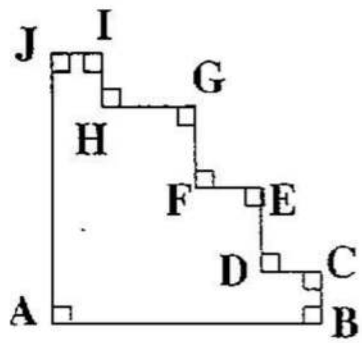
PART I: 10 Minutes NYCIML Contest Three Spring 1999

S99J13. In the game of NIM 500 there are five hundred toothpicks in a pile. Two players alternate turns taking 1,2,3,4,5,6,7, or 8 toothpicks from the pile each time. The *loser* is the one who takes the last toothpick. If you go first, how many toothpicks should you take to guarantee that you win?

S99J14. A set of five positive integers has median 6 and mean 4. There is only one mode in this set, 6. Find the set of five integers.

PART II: 10 Minutes NYCIML Contest Three Spring 1999

S99J15. The figure in the diagram on the right, was cut from a square whose diagonal has length 10. The perimeter of ABCDEFGHIJ can be expressed as $a\sqrt{b}$, where b is a positive prime. Compute the ordered pair (a,b).



S99J16. If x and y are positive integers such that x>y, compute all ordered pairs that satisfy the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$.

PART III: 10 Minutes NYCIML Contest Three Spring 1999

S99J17. The sum of two positive integers is 27. The sum of their cubes is 5103. Compute the two integers.

S99J18. The product $\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\cdot\frac{35}{36}\cdot\ldots\cdot\frac{1999^2-1}{1999^2}$ can be written as a fraction $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute the sum a+b.

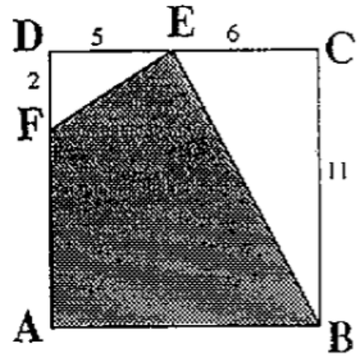
Answers

13. 4	**15.** (20,2)	**17.** 12 and 15
14. {1,1,6,6,6}	**16.** (20,5) and (12,6)	**18.** 9997

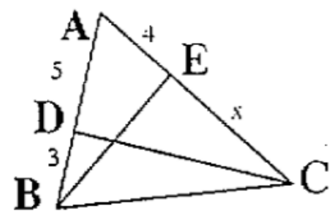
New York City Interscholastic Mathematics League

Solutions

S99J1. The best way to handle this problem is to work backwards: At the very end, you will have 1, 2, 3, 4, or 5 cards from which to choose. Your opponent should have 6 cards to choose from his/her last turn. In order for that to happen, you must have 7, 8, 9, 10, or 11 cards from which to choose. In order for this to happen, your opponent should have 12 cards in front of him/her. This means you must take two cards the first time. <u>Answer:</u> 2



S99J2. It is difficult to find the area of FABE directly. An easier approach is to find the area of square ABCD (which is 121) and subtract the non shaded areas. The area of ΔDEF is 5 and the area of ΔECB is 33. This means that the area of the shaded region is $121 - 38 = 83$. <u>Answer:</u> 83

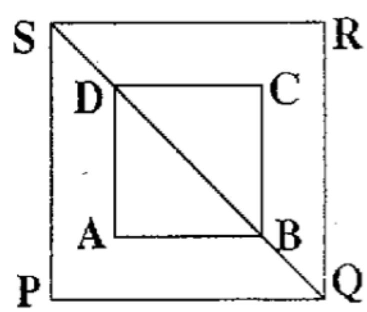


S99J3. There are several approaches to this problem. Probably the simplest one is to factor both numerator and denominator: $\frac{5+10+15+...+200}{7+14+21+...280} = \frac{5(1+2+3...+40)}{7(1+2+3...+40)} = \frac{5}{7}$. This means that a = 5 and b = 7 so that a + b = 12. <u>Answer:</u> 12

S99J4. Let x = EC. Since ΔABE ~ ΔACD, we can set up the following proportion:

$\frac{8}{4} = \frac{x+4}{5}$ This gives x = 6. <u>Answer:</u> 6

S99J5. The probability is the ratio of the area inside square PQRS and outside square ABCD to the area of the square PQRS. Since no measures are given, let SQ = $2\sqrt{2}$. The area of a square whose diagonal has length d is $\frac{1}{2}d^2$. Thus the area of square ABCD = 1 and the area of square PQRS is 4. The area outside of square ABCD within square PQRS is 3. The desired probability is thus $\frac{3}{4}$. <u>Answer:</u> $\frac{3}{4}$



S99J6. In order to simplify the given fraction, we need to factor the numerator and denominator using $x^3 - 1 = (x-1)\cdot(x^2 + x + 1)$ and $x^3 + 1 = (x+1)\cdot(x^2 - x + 1)$. This gives the following:

$$\frac{(2^3-1)(3^3-1)(4^3-1)...(20^3-1)}{(2^3+1)(3^3+1)(4^3+1)...(20^3+1)} = \frac{(2-1)(3-1)(4-1)...(20-1)}{(2+1)(3+1)(4+1)...(20+1)} \cdot \frac{(2^2+2+1)(3^2+3+1)(4^2+4+1)...(20^2+20+1)}{(2^2-2+1)(3^2-3+1)(4^2-4+1)...(20^2-20+1)}$$

$$= \frac{1\cdot2\cdot3\cdot...\cdot19}{3\cdot4\cdot5\cdot...\cdot21} \cdot \frac{7\cdot13\cdot21\cdot31\cdot...\cdot421}{3\cdot7\cdot13\cdot21\cdot...\cdot381} = \frac{2}{20\cdot21} \cdot \frac{421}{3} = \frac{421}{630}$$

<u>Answer:</u> $\frac{421}{630}$

Please note: *Concepts used today will be repeated later this term.*

New York City Interscholastic Mathematics League

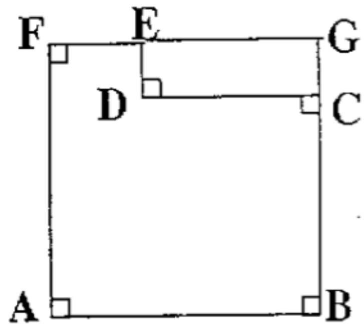
Solutions

S99J7. Working backwards, we aim to have our opponent have 6, 12, 18, 24,.. cards from which to choose. (See S99J1). Following this pattern, our first goal must be to have 198 cards in front of your opponent. This will happen if you choose two cards the first time.

Answer: 2

S99J8. Let s = the price of a sheep. The price of a cow is 12s and the price of an ox is 24s. This gives the equation $s + 12s + 24s = 7400 \rightarrow 37s = 7400$ giving $s = 200$. Thus the price of a sheep is \$200, a cow costs \$2400 and an ox costs \$4800. The cost of ten sheep, three cows and two oxen is \$2000 + \$7200 + \$9600 = \$18,800. Answer: \$18,800

S99J9. In the diagram on the right, extend $\overline{FE}$ and $\overline{BC}$ until they meet at point G. This shows that ABCDEF has the same perimeter as square ABGF, which is 48.



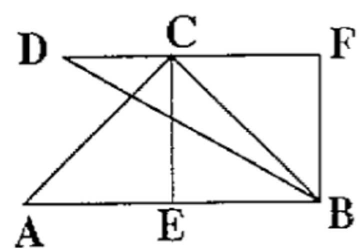
Answer: 48

S99J10. $3^{10}+3^8-10=3^8(3^2+1)-10=3^8(10)-10 = 10(3^8-1) = 10(3^4-1)(3^4+1) = 2\cdot5\cdot80\cdot82 = 2\cdot5\cdot2^4\cdot5\cdot41\cdot2 = 2^6\cdot5^2\cdot41^1$. Thus the needed sum is $2+6+5+2+41+1 = 57$.

Answer: 57

S99J11. $\sqrt{12+\sqrt{x}} = 5 \rightarrow 12+\sqrt{x} = 25 \rightarrow \sqrt{x} = 13$ so $x = 169$. Answer: 169

S99J12. Construct $\overline{CE} \perp \overline{AB}$, extend $\overline{DC}$ to point F so that $\overline{BF} \perp \overline{DF}$. ΔACB is an isosceles right triangle so that altitude $\overline{CE}$ is also a median. The length of the median to the hypotenuse of a right triangle is half the hypotenuse. Thus $CE = \frac{1}{2}AB$. Because parallel lines are equidistant, we have $\overline{CE} \cong \overline{BF}$. We also were given $\overline{AB} \cong \overline{BD}$. Substituting these facts in the previous equation, we get $BF = \frac{1}{2}BD$. This means that ΔBFD is a 30-60-90 triangle with $m\angle D = 30$. Answer: 30



New York City Interscholastic Mathematics League

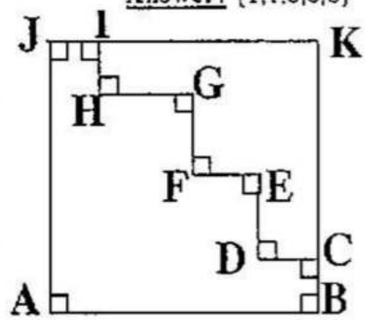
Solutions

S99J13. The best way to handle this problem is to work backwards: At the very end, your opponent will have one toothpick in front of him/her, since you expect him/her to lose. This means that you should have 2,3,4,5,6,7,8, or 9 toothpicks in front of you. This will happen if your opponent has 10 toothpicks in front of him/her. Continuing on this way, your opponent should have 19, 28, 37,... toothpicks from which to choose. These numbers are one more than a multiple of 9. Since 496 is the closest number to 500 that is one more than a multiple of 9, you must choose 4 toothpicks to ensure that you win.

<u>Answer:</u> 4

S99J14. Since 6 is the unique mode, there are at least two 6's. This means the three other numbers in the set have a sum of 8 (since the mean is 4). Although there are several possibilities, the only one that keeps 6 as the median is {1,1,6,6,6}

<u>Answer:</u> {1,1,6,6,6}



S99J15. If the length of the diagonal is 10, each side has length $5\sqrt{2}$. The diagram on the right illustrates that the figure ABCDEFGHIJ has the same perimeter as square ABKJ which is $20\sqrt{2}$. This means that a= 20 and b =2.

<u>Answer:</u> (20,2)

S99J16. Multiplying both sides of the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$ by $4xy$ gives $4y+4x=xy$. This is equivalent to $xy-4x-4y=0$. Adding 16 to both sides gives $xy-4x-4y+16=16$. The left hand side factors nicely giving $(x-4)(y-4)=16$. Since x and y are integers we can test every possibility: <u>Case I:</u> The two factors of 16 are 16 and 1. This gives x = 20 and y =5. <u>Case II:</u> The two factors of 16 are 8 and 2. This gives x = 12 and y =6. There are no other cases since x must be greater than y.

<u>Answer:</u> (20,5) and (12,6)

S99J17. Let x and y = the two positive integers. We have the system $\begin{cases} x+y=27 \\ x^3+y^3=5103 \end{cases}$

In order to solve this system, cube the first equation: $x^3+3x^2y+3xy^2+y^3=27^3$

This is equivalent to $x^3+y^3+3xy(x+y)=27^3 \rightarrow 5103+3xy(27)=27^3 \rightarrow 27(189)+3xy(27)=27^3$

$\rightarrow 189+3xy=27^2 \rightarrow 63+xy=243 \rightarrow xy=180$

From the first equation, we have $y=27-x$ so we can substitute this result into $xy=180$. This gives the quadratic equation $x(27-x)=180 \rightarrow 27x-x^2=180 \rightarrow x^2-27x+180=0 \rightarrow (x-12)(x-15)=0$

Thus $x=12$ or $x=15$. In both cases we get the pair of numbers 12 and 15.

<u>Answer:</u> 12 and 15.

S99J18. $\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\cdot\frac{35}{36}\cdots\frac{1999^2-1}{1999^2}=(1-\frac{1}{3^2})\cdot(1-\frac{1}{4^2})\cdot(1-\frac{1}{5^2})\cdot(1-\frac{1}{6^2})\cdot\ldots\cdot(1-\frac{1}{1999^2})$

$=(1-\frac{1}{3})\cdot(1+\frac{1}{3})\cdot(1-\frac{1}{4})\cdot(1+\frac{1}{4})\cdot(1-\frac{1}{5})\cdot(1+\frac{1}{5})\cdot\ldots\cdot(1-\frac{1}{1999})\cdot(1+\frac{1}{1999})$

$=(\frac{2}{3})\cdot(\frac{4}{3})\cdot(\frac{3}{4})\cdot(\frac{5}{4})\cdot(\frac{4}{5})\cdot(\frac{6}{5})\cdot(\frac{5}{6})\cdot(\frac{7}{6})\cdot\ldots\cdot(\frac{1998}{1999})\cdot(\frac{2000}{1999})$

Notice the pattern: Other than the first and last terms, each term is followed or preceded by its reciprocal. This means that most terms divide out leaving $\frac{2}{3}\cdot\frac{2000}{1999}=\frac{4000}{5997}$. Thus a = 4000 and b = 5997 and a+b = 9997.

<u>Answer:</u> 9997