NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE**

PART I: TIME 10 MINUTES **FALL 1998**

F98B1 Compute the sum of all positive integers less than 1000 which leave a remainder of 2 when divided by 3.

F98B2 A regular hexagon is inscribed in a circle of radius 8. Compute, in terms of π, the sum of the areas of the regions between the hexagon and the circle.

PART II: TIME 10 MINUTES **FALL 1998**

F98B3 Solve for all values of x: $x^2 + |x| - 12 = 0$.

F98B4 Compute the length of the altitude of a regular tetrahedron with edge 6.

PART III: TIME 10 MINUTES **FALL 1998**

F98B5 There are 4 black and 4 white socks in a drawer. If 2 socks are drawn at random, without replacement, compute the probability that they are the same color.

F98B6 If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, express $\log_5 72$ in terms of a and b with no logarithms.

ANSWERS

1. 166,500	3. ± 3	5. $\frac{3}{7}$
2. $64\pi - 96\sqrt{3}$	4. $2\sqrt{6}$	6. $\frac{3a+2b}{1-a}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO**

PART I: TIME 10 MINUTES **FALL 1998**

F98B7 Compute the area of an isosceles trapezoid with sides 6, 8, 8 and 10.

F98B8 Change the base 10 number 1998 to a number in base 3.

PART II: TIME 10 MINUTES **FALL 1998**

F98B9 The graphs of $y = 2\log x$ and $y = \log 2x$ intersect at (a,b). Compute the ordered pair (a,b).

F98B10 Compute the number of positive integral factors of $(12)^4$.

PART III: TIME 10 MINUTES **FALL 1998**

F98B11 A circle is inscribed in an equilateral triangle and a square is inscribed in the circle. Compute the ratio of the area of the square to the area of the triangle.

F98B12 If $x+\frac{1}{x}=4$, compute the value of $x^5+\frac{1}{x^5}$.

ANSWERS

7. $16\sqrt{15}$	9. (2, log 4) or (2, 2log2)	11. $\frac{2\sqrt{3}}{9}$
8. 2202000	10. 45	12. 724

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE**

PART I: TIME 10 MINUTES **FALL 1998**

F98B13 Compute the sum of the digits of the first 100 positive integers.

F98B14 Three fair dice are thrown. Compute the probability that their sum is 5.

PART II: TIME 10 MINUTES **FALL 1998**

F98B15 Compute the sum of the infinite series $0.05 + 0.005 + 0.0005 + \ldots$

F98B16 In how many ways can 5 men and 5 women be seated at a circular table if the men and women alternate?

PART III: TIME 10 MINUTES **FALL 1998**

F98B17 Compute $\sqrt{17 \cdot 18 \cdot 19 \cdot 20 + 1}$ in simplest form.

F98B18 ABC is an equilateral triangle with side 4. $\overline{BC}$ is extended through C to D so that CD = 4. E is the midpoint of $\overline{AB}$, and $\overline{DE}$ intersects $\overline{AC}$ at F. Compute the area of quadrilateral BEFC.

ANSWERS

13. 901	15. $\frac{1}{18}$	17. 341
14. $\frac{1}{36}$	16. 2,880	18. $\frac{8}{3}\sqrt{3}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR**

PART I: TIME 10 MINUTES **FALL 1998**

F98B19 If $i=\sqrt{-1}$, compute $(1+i)^{12}$.

F98B20 How many non congruent triangles with integral sides have a perimeter of 10?

PART II: TIME 10 MINUTES **FALL 1998**

F98B21 Compute the area of the smallest square which passes through (1,0) and (7,0).

F98B22 In triangle ABC, $\tan A=\frac{1}{2}$ *and* $\tan B=\frac{1}{3}$. Compute tan C.

PART III: TIME 10 MINUTES **FALL 1998**

F98B23 Compute $\log_4 \frac{64\cdot 1024}{256}$.

F98B24 If 606, 967 and 1404 are each divided by the positive integer q (q > 1), then they all leave a remainder of r. Compute r.

ANSWERS

19. −64	21. 18	23. 4
20. 2	22. −1	24. 17

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FIVE**

PART I: TIME 10 MINUTES **FALL 1998**

F98B25 If $x - 3$, $x + 2$ and $5x - 1$ are the first three terms of an arithmetic progression, compute x.

F98B26 Working alone, a man can paint a room in 8 hours. Working with his son, they can paint the room in 5 hours. How long would it take the son to paint the room if he worked alone?

PART II: TIME 10 MINUTES **FALL 1998**

F98B27 The area of the top of a box (rectangular solid) is 8, the area of the front is 10, and the area of the side is 12. Compute the volume of the box.

F98B28 Find all ordered pairs of integers (x,y) which satisfy $2^{2x} - 3^{2y} = 55$.

PART III: TIME 10 MINUTES **FALL 1998**

F98B29 Compute the remainder if $x^8 + 1$ is divided by $x - 1$.

F98B30 In a triangle, 2 sides have lengths 10 and 12. The medians to these 2 sides are perpendicular to each other. Compute the length of the third side of the triangle.

ANSWERS

25. 2	27. $8\sqrt{15}$	29. 2
26. $\frac{40}{3}$	28. (3,1)	30. $\frac{2\sqrt{305}}{5}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1998 CONTEST ONE

F98B1 $S = 2 + 5 + ... + 998$. $S = \frac{n}{2}(a + l) = \frac{333}{2}(2 + 998) = 166{,}500$.

F98B2 There are 6 equilateral triangles with side 8 and the area equals $6 \cdot \frac{8^2}{4}\sqrt{3} = 96\sqrt{3}$. The required area is $64\pi - 96\sqrt{3}$.

F98B3 $|x|^2 + |x| - 12 = 0$. $(|x| + 4)(|x| - 3) = 0$. $|x| = -4$ is impossible. Therefore, $|x| = 3 \text{ or } x = \pm 3$.

F98B4 The altitude is a leg of the right triangle whose hypotenuse is 6 and the other leg is $2\sqrt{3}$. Thus, $(2\sqrt{3})^2 + h^2 = 6^2$ or $h = \sqrt{24} = 2\sqrt{6}$.

F98B5 $P = \frac{{}_4C_2 + {}_4C_2}{{}_8C_2} = \frac{6+6}{28} = \frac{12}{28} = \frac{3}{7}$. OR, after one is chosen, there are 7 left, of which 3 will produce a match.

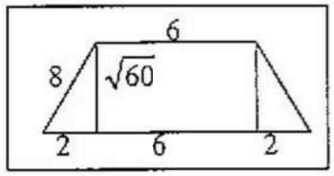
F98B6 $\log_5 72 = x$, $5^x = 72$, $x \log 5 = \log 72$. $x = \frac{\log 72}{\log 5} = \frac{\log 2^3 \cdot 3^2}{\log \frac{10}{2}} = \frac{3\log 2 + 2\log 3}{\log 10 - \log 2}$

$= \frac{3a + 2b}{1 - a}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1998 CONTEST TWO

F98B7 $A=\frac{1}{2}\sqrt{60}(6+10)=16\sqrt{15}$.



F98B8 This can be computed by continuously dividing by 3 and reading the remainders in reverse order. The number is 2202000.

F98B9 2logx = log 2x. $\log x^2=\log 2x$. $x^2=2x$ or $x^2-2x=0$ or $x=2$. (x cannot be 0). (2, 2log2) or (2, log4).

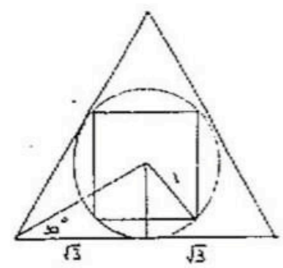
F98B10 $(12)^4=2^8\cdot 3^4$. The number is the product of each exponent increased by one. Thus, $9\times 5=45$.

F98B11 Let the radius of the circle be an arbitrary 1.

Then the square has area $\frac{1}{2}\cdot 2^2=2$, the side

of the triangle is $2\sqrt{3}$ and its area is $\frac{(2\sqrt{3})^2}{4}\sqrt{3}=3\sqrt{3}$.

Ratio is $\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}$.



F98B12 $\left(x+\frac{1}{x}\right)^2=4^2$.

$x^2+2+\frac{1}{x^2}=16$. $\left(x^2+\frac{1}{x^2}\right)^2=(14)^2$. $x^4+2+\frac{1}{x^4}=196$. $x^4+\frac{1}{x^4}=194$

$\left(x+\frac{1}{x}\right)\left(x^4+\frac{1}{x^4}\right)=4\cdot 194$

(a) $x^5+x^3+\frac{1}{x^3}+\frac{1}{x^5}=776$.

$\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)=4\cdot 14$. $x^3+x+\frac{1}{x}+\frac{1}{x^3}=56$

(b) $x^3+\frac{1}{x^3}=52$

Subtracting (a) – (b): $x^5+\frac{1}{x^5}=724$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1998 CONTEST THREE

F98B13 Consider the numbers as 2 digit numbers starting with 00, 01, 02, … . There are 10 zeros, 10 ones, 10 twos, etc. in each place. $10(1+2+3+\ldots+9)=10(45)$ $=450$. $450\times 2=900$. $900+1$ for 100 equals 901.

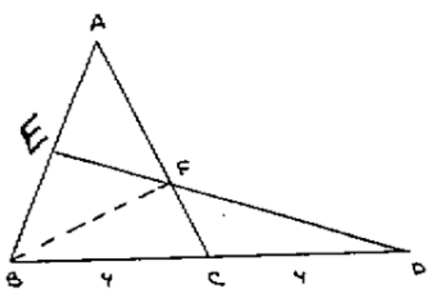
F98B14 The only possibilities are (2,2,1) 3 ways and (3,1,1) 3 ways. $\frac{6}{6^3}=\frac{1}{36}$.

F98B15 This is an infinite geometric progression with a sum of $\frac{\frac{1}{20}}{1-\frac{1}{10}}=\frac{1}{18}$.

F98B16 Seat the men first. Once the first man is seated, there are 4! ways for the others to sit, and then 5! ways for the women to sit. $4!\,5!=24\cdot 120=2880$.

F98B17 If a, b, c and d are consecutive integers, $\sqrt{a\cdot b\cdot c\cdot d+1}=ad+1$. Thus, $17\cdot 20+1=341$.

F98B18 Draw $\overline{BF}$. FC = $\frac{1}{3}AC$ since F is the point of intersection of the medians of ΔABD. $\Delta FBC=\frac{1}{3}\Delta ABC$. $\Delta FEB=\frac{1}{2}\Delta AFB=\frac{1}{3}\Delta ABC$. Quadrilateral BEFC = $\frac{2}{3}\Delta ABC=\frac{2}{3}\cdot 4\cdot\sqrt{3}=\frac{8}{3}\sqrt{3}$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1998 CONTEST FOUR

F98B19 $(1+i)^2 = 1 + 2i - 1 = 2i.$ $(1+i)^{12} = (2i)^6 = 64i^6 = -64$

F98B20 The only possibilities are (4,4,2) and (4,3,3).

F98B21 This segment must be the diagonal. $A = \frac{1}{2} \cdot d^2 = 18$.

F98B22. $\tan C = \tan(180 - [A + B]) = -\tan(A + B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B} =$

$$-\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = -\frac{3+2}{6-1} = -1$$

F98B23 $\log_4 \frac{64 \cdot 1024}{256} = \log_4 64 + \log_4 1024 - \log_4 256 = 3 + 5 - 4 = 4$

F98B24 q must be a factor of the difference since they all leave the same remainder when divided by q. $967 - 606 = 361 = 19 \cdot 19$. $1404 - 967 = 437 = 19 \cdot 23$. Therefore, q = 19 and r = 17.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1998 CONTEST FIVE

F98B25 $x + 2 - (x - 3) = 5x - 1 - (x + 2)$. $x = 2$.

F98B26 The man paints $\frac{5}{8}$ of the room, the son must paint $\frac{3}{8}$ of the room. $\frac{5}{x} = \frac{3}{8}$.

$x = \frac{40}{3}$.

F98B27 $ab = 8$, $bc = 10$ and $ac = 12$. Taking the product,
$a^2b^2c^2 = 960$, $abc = \sqrt{960}$ *or* $8\sqrt{15}$.

F98B28 $(2^x + 3^y)(2^x - 3^y) = 55$. Thus either $2^x + 3^y = 55$ *and* $2^x - 3^y = 1$ which does not have a solution or $2^x + 3^y = 11$ *and* $2^x - 3^y = 5$ which yields $x = 3$ and $y = 1$.

F98B29 Using long division, the remainder is 2, or using the remainder theorem, the remainder when f (x) is divided by x – a is f (a). f (1) = 2.

F98B30

$$4x^2 + y^2 = 25$$
$$x^2 + 4y^2 = 36$$
$$5x^2 + 5y^2 = 61$$
$$x^2 + y^2 = \frac{61}{5}$$
$$4x^2 + 4y^2 = \frac{244}{5}$$
$$\sqrt{4x^2 + 4y^2} = \sqrt{\frac{244}{5}}$$
$$= \frac{2\sqrt{305}}{5}$$