New York City Interscholastic Math League

Senior A Division | **Contest Number One**

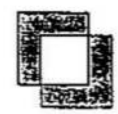
Part I — *Fall, 1998* — *Contest 1* — *Time: 10 Minutes*

F98S1 How many three-digit positive integers can be written if all three digits are different prime numbers?

F98S2 Compute the value of $\frac{\sqrt{3}}{3+\sqrt{8}}+\frac{\sqrt{3}}{\sqrt{8}+\sqrt{7}}+\frac{\sqrt{3}}{\sqrt{7}+\sqrt{6}}+\frac{\sqrt{3}}{\sqrt{6}+\sqrt{5}}+\frac{\sqrt{3}}{\sqrt{5}+2}$.

Part II — *Fall, 1998* — *Contest 1* — *Time: 10 Minutes*

F98S3 The two squares shown are congruent and their common region is also a square. The length of each side is integral. The sum of the areas of the shaded regions is 82. Compute the length of one side of one of the congruent squares.



F98S4 The hypotenuse of a right triangle exceeds a leg by 1. The radius of the inscribed circle is 5. Compute the length of the hypotenuse.

Part III — *Fall, 1998* — *Contest 1* — *Time: 10 Minutes*

F98S5 How many positive integers less than 1998 leave a remainder of 5 when divided by all of the following: 6, 10, and 15?

F98S6 Compute all values of A in the interval $180° < A° < 270°$ that satisfy

$$4\sin^3 A - 3\sin A = \tfrac{1}{2}.$$

Answers:

Answers:	F98S1	24
	F98S2	$\sqrt{3}$
	F98S3	21
	F98S4	61
	F98S5	66
	F98S6	190, 230

New York City Interscholastic Math League

Senior A Division | **Contest Number Two**

Part I | *Fall, 1998* | *Contest 2* | *Time: 10 Minutes*

F98S7 Define the sequence $\{a_n\}$ thusly: $a_1 = 0$, $a_2 = 1998$, and $a_k = a_{k-1} - a_{k-2}$ for integral $k \geq 2$. Compute a_{1998}.

F98S8 A rectangular solid is 14 units long and 2 units wide. If the height h and the diagonal d are both integral in measure, compute all possible values of (d,h).

Part II | *Fall, 1998* | *Contest 2* | *Time: 10 Minutes*

F98S9 Compute x such that $\sqrt{-(2x+7)^2}$ is real.

F98S10 Compute all values of x such that

$$3^{2x+1} + 3^2 = 3^{x+3} + 3^x.$$

Part III | *Fall, 1998* | *Contest 2* | *Time: 10 Minutes*

F98S11 Compute the value of y that satisfies $\log_5 y = \left(-2\log_5 \sqrt{2}\right)\left(1 + \log_4 144\right)$

F98S12 The area of an acute triangle is K and two of its legs are a and b. In simplest terms, express the length of the third side in terms of a, b, and K.

Answers:

F98S7	-1998
F98S8	(51,49), (27,23), (15,5)
F98S9	-3.5
F98S10	$-1, 2$
F98S11	$^1/_{24}$
F98S12	$\sqrt{a^2 + b^2 - 2\sqrt{a^2b^2 - 4K^2}}$

New York City Interscholastic Math League

Senior A Division | **Contest Number Three**

Part I *Fall, 1998* *Contest 3* *Time: 10 Minutes*

F98S13 If $2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14 = n\cdot8!$, compute n.

F98S14 For members of the James Garfield HS Math Team, the probability of answering this question correctly is 80% for Mary, 75% for Todd, and n% for Lincoln. If the probability that exactly two of these three mathletes answer this question correctly is 45%, compute n.

Part II *Fall, 1998* *Contest 3* *Time: 10 Minutes*

F98S15 If $2a+3b+2c=4n$, $3a-b+3c=n+2$, and $-a+2b-c=n$, compute the value of n.

F98S16 Compute all values of x such that

$$(x^2-9x+14)^2+(x^2-8x+15)^2 = (2x^2-17x+29)^2$$

Part III *Fall, 1998* *Contest 3* *Time: 10 Minutes*

F98S17 Compute the value of $\dfrac{\sqrt{7+2\sqrt{10}}+\sqrt{7-2\sqrt{10}}}{\sqrt{7+2\sqrt{10}}-\sqrt{7-2\sqrt{10}}}$.

F98S18 If $\tan^2(180°-x)+\sec(180°+x)=11$, where x is in degrees, compute all values of $\cos x$.

Answers:

F98S13	16
F98S14	60
F98S15	7
F98S16	2, 3, 5, 7
F98S17	$\frac{1}{2}\sqrt{10}$
F98S18	$\frac{1}{4}, -\frac{1}{3}$

New York City Interscholastic Math League

Senior A Division | **Contest Number Four**

Part I *Fall, 1998* *Contest 4* *Time: 10 Minutes*

F98S19 If ${}_nP_4 = 20 \cdot {}_{n-1}C_2$, compute n.

F98S20 If $a + b = 5$, compute the minimum value of $a^3 + b^3$.

Part II *Fall, 1998* *Contest 4* *Time: 10 Minutes*

F98S21 If the graph of $y = x^2 - 6x + 15$ is reflected over the line $y = 4$, the image is the graph of $y = ax^2 + bx + c$. Compute the ordered triple (a, b, c).

F98S22 One side of a triangle is 20 and the perimeter is 72. Compute the maximum area of the triangle.

Part III *Fall, 1998* *Contest 4* *Time: 10 Minutes*

F98S23 Sari scores 173 points on a certain contest of 50 questions. She receives 8 points for each correct answer, 3 points for each omitted answer and 0 points for each wrong answer. What is the maximum number of wrong answers that she could have?

F98S24 Compute the least value of x such that $\sqrt[3]{\frac{x-62}{x}} + 3 \cdot \sqrt[3]{\frac{x}{x-62}} = \frac{28}{5}$.

Answers:	F98S19	5
	F98S20	$31\frac{1}{4}$ or $^{125}/_4$ or 31.25
	F98S21	$(-1, 6, -7)$
	F98S22	240
	F98S23	24
	F98S24	$-\frac{1}{2}$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

SENIOR A DIVISION | **CONTEST NUMBER FIVE**

PART I ***FALL, 1998*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F98S25 A fraction can be written as either $.\overline{7}$ in base a or $.\overline{3}$ in base b, where a and b are positive integers. In simplest terms, express b in terms of a. [note: $.\overline{n}$ represents the repeating decimal $0.nnn\ldots$]

F98S26 After simplifying, how many terms should be written in the expansion of the expression $(a_1 + a_2 + a_3 + \ldots + a_{100})^2$?

PART II ***FALL, 1998*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F98S27 When filled to capacity, each of two cylinders holds equal amounts of liquid. One cylinder has a height of 12 and a circumference of 12π. The other cylinder has a height of h and a circumference of 8π. Compute h.

F98S28 For two values of a, the graph of $y = x^2 - 4x + 3$ intersects the graph of $y = ax - 15$ such that the abscissa of one point of intersection is twice the abscissa of the second point of intersection. Compute both values of a.

PART III ***FALL, 1998*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F98S29 Every natural number is in this table in order, filling every position. Each row "goes" in the reverse direction from the preceding row. What letter heads the column that contains 1998?

A	B	C	D	E	F	G
			1	2	3	4
11	10	9	8	7	6	5
12	13	14	15	16	17	18
25	24	23	22	21	20	19
26	27	28	29	30	31	32

etc.

F98S30 In ΔABC, D is the midpoint of side $\overline{BC}$, and points E and F trisect median $\overline{AEFD}$, $\overline{BE}$ and $\overline{BF}$ meet side $\overline{AGHC}$ in points G and H respectively. If AG = 10, compute the ordered pair (GH, HC).

ANSWERS:

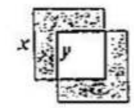
F98S25	$b = \frac{3a+4}{7}$
F98S26	5050
F98S27	27
F98S28	5, −13 (both required)
F98S29	B
F98S30	(15,25)

SOLUTIONS

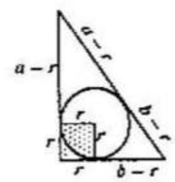
F98S1 **Answer: 24** There are 4 prime digits: 2, 3, 5, and 7.Then there are $4\cdot3\cdot2 = 24$ permutations.

F98S2 **Answer: $\sqrt{3}$** In each fraction rationalize the denominator; each denominator becomes 1. The numerators then become $\left(\sqrt{3}\right)\left[\left(3-\sqrt{8}\right)+\left(\sqrt{8}-\sqrt{7}\right)+\left(\sqrt{7}-\sqrt{6}\right)+\left(\sqrt{6}-\sqrt{5}\right)+\left(\sqrt{5}-2\right)\right]=\sqrt{3}$.

F98S3 **Answer: 21** From either of the congruent squares: $x^2 - y^2 = \frac{1}{2}(82) = 41$. Then $(x-y)(x+y) = 1\times41$. The system $x + y = 41$ and $x - y = 1$ produces (21,20).



F98S4 **Answer: 61** Using r = radius, a, b = legs and c = hypotenuse: each side of the shaded square is r, so the other segments of the legs are $a - r$ and $b - r$. Because tangents from a point to a circle are congruent, the segments of the hypotenuse are $a-r$ and $b-r$. Then $c = (a-r) + (b-r)$ and $c = b + 1$ produces $2r = a - 1$. $\therefore \frac{1}{2}(a-1) = r = 5$, so that $a = 11$. Since $a^2 + (c-1)^2 = c^2$, $c = \frac{1}{2}(a^2+1) = \frac{1}{2}(11^2+1) = 61$.



F98S5 **Answer: 66** The set of common multiples of 6, 10, and 15 is equivalent to the set of multiples of 30. The question is tantamount to counting the number of positive multiples of 30 which are less than 1998 – 5. The largest multiple of 30 less than 1993 is 1980. $1980 \div 30 = 66$.

F98S6 **Answer: 190, 230**

$$\begin{aligned}
4\sin^3 A - 3\sin A &= \tfrac{1}{2}\\
3\sin A - 4\sin^3 A &= -\tfrac{1}{2}\\
\text{But } 3\sin A - 4\sin^3 A &= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A\\
&= 2\sin A(1-\sin^2 A) + \sin A(1-2\sin^2 A)\\
&= 2\sin A\cos^2 A + \sin A(\cos 2A)\\
&= \cos A\sin 2A + \sin A\cos 2A\\
&= \sin(2A + A)\\
&= \sin 3A, \text{ which} = -\tfrac{1}{2}.
\end{aligned}$$

Thus, $3A = 210 + 360k$ or $3A = 330 + 360k$, k = an integer.
Then $A = 70 + 120k$ or $A = 110 + 120k$. That is, $A = 70, 190, 310, \ldots$ or $A = 110, 230, 340, \ldots$, of which only 190 and 230 are in the desired range.

SOLUTIONS

F98S7 **Answer: –1998**

k	1	2	3	4	5	6	7	8	9	10	11	12
a_k	0	1998	1998	0	-1998	-1998	0	1998	1998	0	-1998	-1998

Note that a_k repeats in cycles of 6: $a_1 = a_7$, $a_2 = a_8$, $a_3 = a_9$, $a_4 = a_{10}$, $a_5 = a_{11}$, and $a_6 = a_{12} = a_{6n}$, for integral n. 1998 is a multiple of 6, so $a_{1998} = a_6 = -1998$.

F98S8 **Answer: (51,49), (27,23), (15,5)** $14^2 + 2^2 + h^2 = d^2 \Rightarrow d^2 - h^2 = 200 \Rightarrow (d-h)(d+h) = 200$. Then $d-h$ and $d+h$ are cofactors of 200, with $d-h < d+h$. Thus the six equations $d-h = 1$, 2, 4, 5, 8, or 10 are paired with the six equations $d+h = 200$, 100, 50, 40, 25, or 20, respectively. Solve each system of equations to find 3 pairs of ordered positive integers and 3 pairs of ordered fractions.

F98S9 **Answer: –3.5** $-(2x+7)^2 \le 0$, but to be real, radicand must ≥ 0. $\therefore -(2x+7)^2 = 0$, so $x = -3.5$.

F98S10 **Answer: –1, 2** $3^{2x+1} + 3^2 = 3^{x+3} + 3^x \to 3(3^{2x}) + 3^2 = 3^3(3^x) + 3^x$.

Let $a = 3^x$. Then $3a^2 + 9 = 3^3a + a \to 3a^2 - 28a + 9 = 0 \to a = \frac{1}{3}, 9$.

Thus $3^x = 3^{-1}$ or $3^x = 3^2$.

F98S11 **Answer: $\frac{1}{24}$** $\log_5 y = -2\log_5\sqrt{2}(1+\log_4 144) = -\log_5 2(1+\log_4 144) = -\log_5 2 - \log_5 2 \cdot \log_4 144$

$$= -\frac{\log 2}{\log 5} - \frac{\log 2}{\log 5}\cdot\frac{\log 144}{2\ \log 2} = -\frac{\log 2}{\log 5} - \frac{2\ \log 12}{2\ \log 5} = -\frac{\log 24}{\log 5} = \log_5 24^{-1}$$

$$\therefore y = \frac{1}{24}$$

F98S12 **Answer: $\sqrt{a^2+b^2-2\sqrt{a^2b^2-4K^2}}$**. $K = \frac{1}{2}ab\sin\theta \to \sin\theta = \frac{2K}{ab}$. Then, since $\theta < 90°$,

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{4K^2}{a^2b^2}} = \frac{1}{ab}\sqrt{a^2b^2-4K^2}$. Finally, if x represents the length of the third side of the triangle, by the Law of Cosines,

$$x^2 = a^2 + b^2 - 2ab\left(\frac{1}{ab}\sqrt{a^2b^2-4K^2}\right) \to x = \sqrt{a^2+b^2-2\sqrt{a^2b^2-4K^2}}.$$

SOLUTIONS

F98S13 **Answer: 16** Noting that $2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14 = 2^7\cdot7!$ and that $8! = 8\cdot7!$, the equation becomes $2^7\cdot7! = 8\cdot7!n$. This implies that $2^7 = 2^3n$. Then $n = 2^4 = 16$.

F98S14 **Answer: 60** The probability that Lincoln is correct is $\frac{n}{100}$ and the probability that Lincoln is incorrect is $\frac{100-n}{100}$. Since P(exactly two of these three) = $P(A\wedge B\wedge\sim C) + P(A\wedge\sim B\wedge C) + P(\sim A\wedge B\wedge C)$, then $\frac{45}{100} = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{100-n}{100} + \frac{4}{5}\cdot\frac{1}{4}\cdot\frac{n}{100} + \frac{1}{5}\cdot\frac{3}{4}\cdot\frac{n}{100}$. Thus $n = 60$, and the probability is 60%.

F98S15 **Answer: 7** Let $d = a + c$. Then the three equations become $2d + 3b = 4n$, $3d - b = n+2$, and $-d + 2b = n$. Pair equations to produce two different expressions for d in terms of n, and then equate. Thus, equations 1 and 2 produce $d = {}^{7n+6}/_{11}$ and equations 2 and 3 produce $d = {}^{3n+4}/_{5}$. Equating produces $n = 7$. [Note: $d = 5$; $b = 6$, and a and c are any two numbers whose sum is 5.]

F98S16 **Answer: 2, 3, 5, 7** Since $2x^2 - 17x + 29$ is the sum of $x^2 - 9x + 14$ and $x^2 - 8x + 15$, the form of the equation is $A^2 + B^2 = (A + B)^2$, which implies that $AB = 0 \rightarrow A = 0$ or $B = 0$. Thus $x^2 - 9x + 14 = 0$ or $x^2 - 8x + 15 = 0 \rightarrow x = 2, 7$ or $x = 3, 5$.

F98S17 **Answer: $\frac{1}{2}\sqrt{10}$** Suppose $\sqrt{x+2\sqrt{y}} = \sqrt{a}+\sqrt{b}$. Square both sides: $x+2\sqrt{y} = a+b+2\sqrt{ab} \rightarrow$ $x = a+b$ and $y = ab$. $\therefore \sqrt{7+2\sqrt{10}} = \sqrt{5}+\sqrt{2}$, and $\sqrt{7-2\sqrt{10}} = \sqrt{5}-\sqrt{2}$. Thus the numerator is $\left(\sqrt{5}+\sqrt{2}\right)+\left(\sqrt{5}-\sqrt{2}\right) = 2\sqrt{5}$, and the denominator is $\left(\sqrt{5}+\sqrt{2}\right)-\left(\sqrt{5}-\sqrt{2}\right) = 2\sqrt{2}$. The fraction is $\sqrt{\frac{5}{2}} = \frac{1}{2}\sqrt{10}$.

F98S18 **Answer: $\frac{1}{4}, -\frac{1}{3}$** Noting that $\tan(180° - x) = -\tan x$ and that $\sec(180° + x) = -\sec x$, we have $(-\tan x)^2 + (-\sec x) = \tan^2 x - \sec x = (\sec^2 x - 1) - \sec x = 11$. Then the solution of $\sec^2 x - \sec x - 12 = 0$ is $\sec x = 4, -3$. Thus $\cos x = 1/4$ or $\cos x = -1/3$.

SOLUTIONS

F98S19 **Answer: 5** $n(n-1)(n-2)(n-3) = 20[(½)(n-1)(n-2)] \rightarrow n(n-3) = 10 \rightarrow n = -2$ or 5. Since $n \geq 4$, $n = 5$.

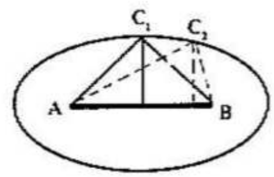
F98S20 **Answer: 31.25 or 125/4**

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\ &= (a+b)(a^2 + 2ab + b^2 - 3\,ab) \\ &= (a+b)[(a+b)^2 - 3ab] \\ &= 5\,[25 - 3ab]. \end{aligned}$$

To minimize $a^3 + b^3$, maximize ab = a$(5 - a)$. Then the maximum point of the parabola whose equation is $y = 5a - a^2$ occurs when $a = 2.5 = b$. Then $a^3 + b^3 = 31.25$.

F98S21 **Answer: (–1, 6, –7)** $y = x^2 - 6x + 15 \rightarrow y = (x-3)^2 + 6$. This graph opens upward and its minimum point is (3,6). After reflection over $y = 4$, the image opens downward and the maximum point is (3,2). Thus the equation of the image is $y = -(x-3)^2 + 2 \rightarrow y = -x^2 + 6x - 7$.

F98S22 **Answer: 240** In ΔABC, AB = 20. Then AC + CB = 52. Form the ellipse of all possible positions of C by using A and B as focal points. The altitude to base $\overline{AB}$ is a maximum if ΔABC is isosceles. Thus the legs are each 26, the altitude is 24, and the area is 240.



F98S23 **Answer: 24** Suppose Sari has a answers omitted, b incorrect, and c correct. Then $a + b + c = 50$ and $3a + 0b + 8c = 173$. Eliminating a yields $5c - 3b = 23$. Since $b \geq 0$, integral $c \geq 5$. Now make a table: assign the values 5, 6, 7, etc. to c; then compute b and a, using $b = {}^{5c-23}/_3$, and $a = 50 - b - c$. Choose only those values of c that yield integral b. The first solution encountered is $c = 7$, $b = 4$ and $a = 39$. Because $c = {}^3/_5 b + {}^{23}/_5$ has a slope of ${}^3/_5$, all integral (c, b, a) are of the form $(7 + 3k, 4 + 5k, 39 - 8k)$ for some integer k. Observe that b is maximal (as is c) when a is minimal. Thus, $a = 39 - 8k \geq 0$, which implies that the maximum value for k is 4. Then the maximum value for $b = 4 + 5k = 4 + 5(4) = 24$.

F98S24 **Answer: –½** Let $n = \sqrt[3]{\frac{x-62}{x}}$. Then $n + \frac{3}{n} = \frac{28}{5} \rightarrow 5n^2 - 28n + 15 = 0 \rightarrow n = 5$ or 0.6.

If $n = 5$, $x = -½$. If $n = 0.6$, $x > 0$. Thus the least value of x is –½.

SOLUTIONS

F98S25 **Answer:** $\frac{3a+4}{7}$

$$S_a = \frac{7}{a} + \frac{7}{a^2} + \frac{7}{a^3} + \cdots = \frac{\frac{7}{a}}{1-\frac{1}{a}} = \frac{7}{a-1} \text{ and}$$

$$S_b = \frac{3}{b} + \frac{3}{b^2} + \frac{3}{b^3} + \cdots = \frac{\frac{3}{b}}{1-\frac{1}{b}} = \frac{3}{b-1}$$

$$\therefore \frac{7}{a-1} = \frac{3}{b-1} \rightarrow b = \frac{3a+4}{7}$$

F98S26 **Answer: 5050** Each of the 100 variables must combine with each of the remaining 99 variables. Because of commutation, there will be ${}_{100}C_2 = \frac{1}{2}(100)(99) = 4950$ terms. Also: each variable is squared once. This provides 100 more terms. In all, there are 4950 + 100 = 5050 terms.

F98S27 **Answer: 27** $V_1 = V_2 = \pi r^2 h$. Then $\pi \times 6 \times 6 \times 12 = \pi \times 4 \times 4 \times h$. Then $h = 27$.

F98S28 **Answer: 5, –13** Equate: $x^2 - 4x + 3 = ax - 15$. Then $x^2 - (a+4)x + 18 = 0$. Let x_1 and x_2 be the roots. Since $x_1 = 2x_2$, $x_1 x_2 = 2x_2^2 = 18$, and the roots are either 3 and 6, or –3 and –6. Then $a + 4 = 3 + 6$ so that $a = 5$, or $a + 4 = -3 + -6$ so that $a = -13$.

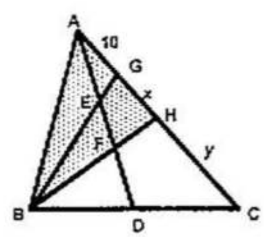
F98S29 **Answer: B** Don't knock yourself out — set up the following table:

A	B	C	D	E	F	G	G	F	E	D	C	B	A
			1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23	24	25 . . .

Hence, 1998 is congruent to 10 (mod 14), which implies that both are in the same column, namely, the second **B**.

F98S30 **Answer: (15, 25)** [1] Since $\overline{BH}$ trisects $\overline{AD}$, $\overline{BH}$ is a median of $\triangle ABC$. $\Rightarrow 10 + x = y$.

[2] Use mass point geometry on $\triangle ABH$. BF:FH = 2:1, so we can hang weights of 1 at B, 2 at H, and 3 at F. Since AE = EF, hang 3 at A. $\Rightarrow$ AG:GH = 2:3 = 10:x $\Rightarrow x = 15$ and $y = 10 + 15 = 25$.



January 24, 1999

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1998 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Senior A	F98S5	67
	F98S9	It should have read "Compute all real x"
Junior	F98J12 was eliminated	It should have read "exactly 3 shaded small triangles" The resulting triangles includes the original triangle. Answer: 4/27
	F98J14	2500

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML