New York City
Interscholastic
Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 1998**

PART I: 10 Minutes **NYCIML Contest One** **Fall 1998**

F98J1. Jack went to work early one morning and was able to drive at an average of 60 miles per hour. Returning home, he rode over the same route but hit traffic. If he averaged 30 miles per hour on the way home, compute his average speed for the round trip to and from work.

F98J2. Compute the value of $\sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$.

PART II: 10 Minutes **NYCIML Contest One** **Fall 1998**

F98J3. Circles A and B intersect at points P and Q. If point B is on Circle A, point A is on Circle B, and AP = 6, compute the area of quadrilateral APBQ.

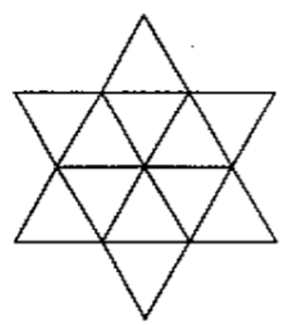
F98J4. On a trip, some friends hiked one mile east, then one mile northeast and then one mile east. The distance from the starting point to the ending point can be written in the form $\sqrt{a+b\sqrt{c}}$, where a, b, and c are positive integers and c is not divisible by the square of any prime number. Compute the value of a+b+c.

PART III: 10 Minutes **NYCIML Contest One** **Fall 1998**

F98J5. Compute the value of ab in the following system of equations:

$$\begin{cases} a + b + \sqrt{a+b} = 20 \\ a^2 + b^2 = 130 \end{cases}$$

F98J6. The "Star of David" drawn on the right has been broken into many triangles by connecting all points of intersection. Compute the total number of triangles in the figure.



Answers

1. 40	**3.** $18\sqrt{3}$	**5.** 63
2. 4	**4.** 9	**6.** 20

New York City Interscholastic Mathematics League

JUNIOR DIVISION　　CONTEST NUMBER TWO　　FALL 1998

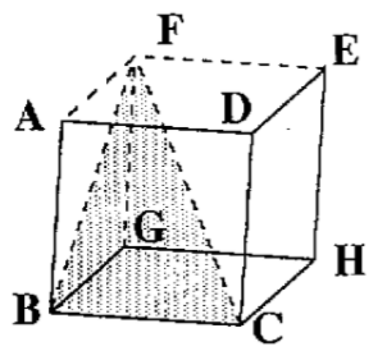
PART I: 10 Minutes　　**NYCIML Contest Two**　　**Fall 1998**

F98J7. Jack went to work early one morning and drove at an average of 60 miles per hour. Returning home, he rode over the same route but hit a traffic jam. If he averaged 30 miles per hour *for the round trip*, compute his average speed driving home.

F98J8. Suppose $x^2 - x - N$ can be factored as the product of two linear factors with integer coefficients. Compute the number of possible values of N if N is a two-digit positive integer.

PART II: 10 Minutes　　**NYCIML Contest Two**　　**Fall 1998**

F98J9. In the diagram on the right, a cube is drawn. If the area of ΔBFC is $32\sqrt{2}$, compute the surface area of the cube.

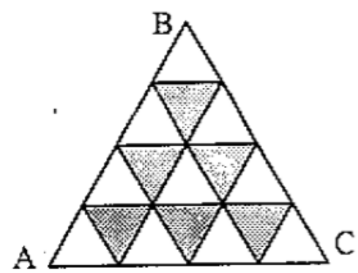


F98J10. In ΔABC, medians $\overline{AP}$ and $\overline{BQ}$ are drawn. If $\overline{AP} \perp \overline{BQ}$, AP = 18, and BQ = 14, compute the area of ΔABC.

PART III: 10 Minutes　　**NYCIML Contest Two**　　**Fall 1998**

F98J11. If $xy = 12$ and $x^2y + xy^2 + x + y = 104$, compute the value of $x^2 + y^2$.

F98J12. Equilateral ΔABC is shown on the right. It was subdivided into smaller equilateral triangles by dividing each side into four congruent segments and connecting all points that result in line segments parallel to a side of ΔABC. If a resulting triangle is chosen at random, compute the probability that the triangle chosen has three shaded small triangles.



Answers

7. 20	9. 384	11. 40
8. 7	10. 168	12. $\frac{3}{26}$

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1998**

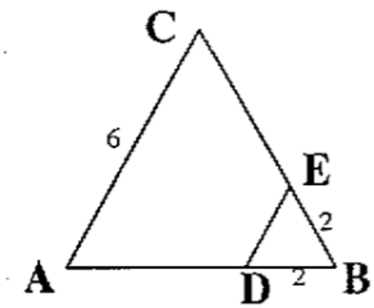
PART I: 10 Minutes NYCIML Contest Three Fall 1998

F98J13. Steve has \$3.80 in coins on him. If he has only dimes and quarters and has twenty coins in all, compute the ordered pair (d,q) where d is the number of dimes and q is the number of quarters he has.

F98J14. Equilateral ΔABC whose side has length 100 centimeters is subdivided into smaller equilateral triangles by dividing each side into fifty congruent segments and connecting only those points that create line segments parallel to a side of the original triangle. Compute the number of equilateral triangles that result having length 2 centimeters on a side.

PART II: 10 Minutes NYCIML Contest Three Fall 1998

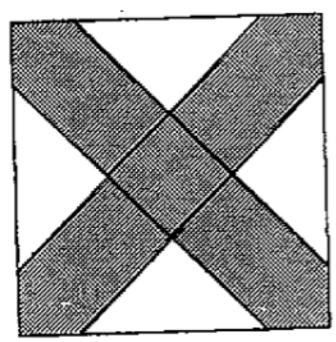
F98J15. In the diagram on the right, equilateral ΔABC is shown with AC = 6. Points D and E are chosen so that DB = EB = 2. Compute the area of quadrilateral ADEC.



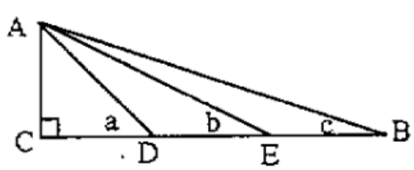
F98J16. Jack went to work early one morning and drove at an average 20 miles per hour faster than he averaged driving home over the same route. If he averaged 37.5 miles per hour *for the round trip*, compute his average speed driving home.

PART III: 10 Minutes NYCIML Contest Three Fall 1998

F98J17. The diagram on the right depicts a square warehouse 600 feet long on each side. There are two walkways (as shaded in the diagram), each symmetrical about a diagonal. There are 400 feet on each side of the square in between the walkways. Compute the shaded area.



F98J18. In the diagram below, AC = 1 and BC = 3 in right ΔABC. D and E are trisection points of $\overline{CB}$. If a = m$\angle$ADC, b = m$\angle$AEC, and c = m$\angle$ABC, compute the value of a+b+c, where a, b, and c are measured in degrees.



Answers

13. (8,12)	15. $8\sqrt{3}$	17. 200,000
14. 10000	16. 30	18. 90

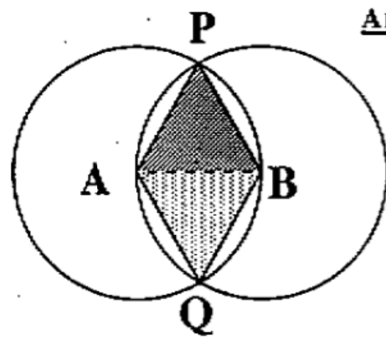
New York City Interscholastic Mathematics League

Solutions

F98J1. **Method I:** Let x = the distance Jack had to travel to work. His average speed is $\frac{\text{Total Distance}}{\text{Total Time}}$. This gives $\frac{2x}{\frac{x}{60}+\frac{x}{30}} = \frac{120x}{x+2x} = 40$.

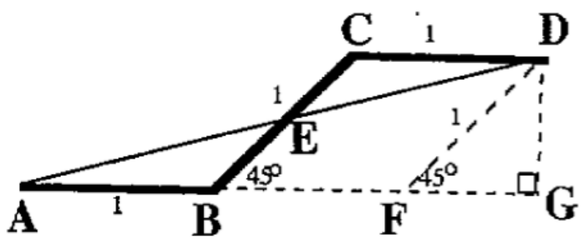
Method II: The average speed on a round trip, following the same route is the harmonic mean of the two speeds. This gives $\frac{2}{\frac{1}{60}+\frac{1}{30}} = \frac{120}{3} = 40$ **Answer:** 40

F98J2. Let $x = \sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$. $x^2 = 12 + x \rightarrow x = -3$ or $x = 4$. Since x is obviously positive, the only reasonable answer is x = 4. **Answer:** 4



F98J3. ΔAPB and ΔAQB must be equilateral. The area of an equilateral triangle whose side has length "s" is given by the formula $A = \frac{s^2\sqrt{3}}{4}$. Thus each of these triangles has area $9\sqrt{3}$. The total required area is therefore $18\sqrt{3}$

Answer: $18\sqrt{3}$



F98J4. In the diagram on the right, $\overline{AB}$ is extended to point G so that $\overline{DG} \perp \overline{AB}$. We need to find AD. It is easy to verify that BCDF is a rhombus, so BF = FD = 1. Since ΔFGD is an isosceles right triangle, FG = GD $= \frac{\sqrt{2}}{2}$. We now use the Pythagorean Theorem on right ΔAGD: $(2 + \frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = AD^2$.

This gives $4 + 2\sqrt{2} + \frac{1}{2} + \frac{1}{2} = AD^2$. This gives $AD = \sqrt{5 + 2\sqrt{2}}$, so a=5, b=c=2. **Answer:** 9.

F98J5. Let $x = \sqrt{a+b}$. This gives $x^2 = a + b$. The first equation becomes $x^2 + x = 20$ which has roots $x = -5$ and 4. Obviously x cannot be negative so x = 4. Since $a+b = x^2$, we have a + b = 16. We now have the system: $\begin{cases} a + b = 16 \\ a^2 + b^2 = 130 \end{cases}$ $\rightarrow$ Squaring the first equation gives $\begin{cases} a^2 + 2ab + b^2 = 256 \\ a^2 + b^2 = 130 \end{cases}$
Subtracting gives 2ab = 126 so that ab = 63. **Answer:** 256

F98J6. In order to count systematically, first count the smallest triangles. There are 12 of them. The next sized triangle has two small segments on a side. There are 6 of them. The biggest triangles consist of three small segments on a side and there are two of them. All together there are 20 triangles.

Answer: 20

Please note: Concepts used today will be repeated later this year.

New York City Interscholastic Mathematics League

Solutions

F98J7. Let x = Jack's average speed coming home. The average speed on a round trip, following the same route is the harmonic mean of the two speeds. This gives the equation $\frac{2}{\frac{1}{60}+\frac{1}{x}} = 30$. $\rightarrow$ $\frac{120x}{x+60} = 30$, which gives $x = 20$ **Answer:** 20

F98J8. Let $x^2 - x - N = (x-a)(x+b)$, where a and b are integers. Note that $|a|$ and $|b|$ must be consecutive integers with $|a| > |b|$ and $N = ab$. The following table lists all possibilities for a and b.

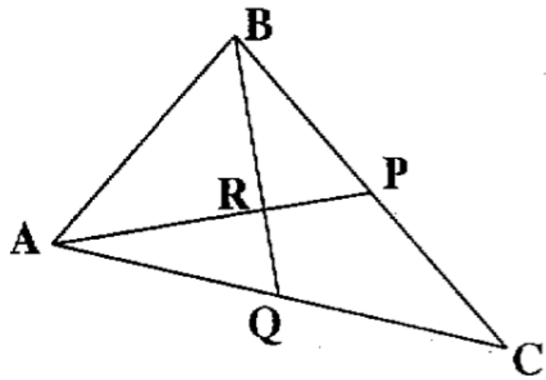
a	2	3	4	5	6	7	8	9	10
b	1	2	3	4	5	6	7	8	9
ab	2	6	12	20	30	42	56	72	99

If $a > 10$, ab will not be a two digit number. Thus there are 7 possible values of N. **Answer:** 7

F98J9. Let x = the length of each edge of the cube. Since ΔBGF is an isosceles right triangle, $BF = x\sqrt{2}$. Thus the area of ΔBFC is $\frac{1}{2}x^2\sqrt{2}$. This gives the equation $\frac{1}{2}x^2\sqrt{2} = 32\sqrt{2}$. This gives $x=8$ and the surface area of the cube is $64 \cdot 6 = 384$. **Answer:** 384

F98J10. In the diagram on the right, we show the medians meeting at point R. The medians of a triangle meet at a point $\frac{2}{3}$ the distance from a vertex to midpoint of the opposite side. This means that $AR = 12$. The area of $\Delta AQB = \frac{1}{2} \cdot 14 \cdot 12 = 84$. Since a median cuts a triangle into two triangles of equal area, then the area of ΔQBC is also 84. Thus, the area of ΔABC is 168.

Answer: 168



F98J11. $x^2y + xy^2 + x + y = 104 \rightarrow xy(x+y) + x + y = 104 \rightarrow 12(x+y) + x + y = 104$

This means that $13(x+y) = 104$ and $x+y = 8$. This gives $(x+y)^2 = 64 \rightarrow x^2 + 2xy + y^2 = 64$ so that $x^2 + 24 + y^2 = 64 \rightarrow x^2 + y^2 = 40$ **Answer:** 40

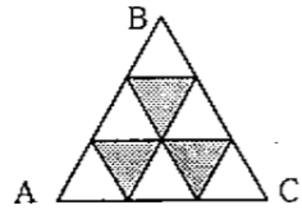
F98J12. First count the smallest triangles. There are 16 of them. Now count the triangles two units on a side. There are 6 of them. There are 3 triangles, three units on a side and one triangle four units on a side. Thus there are 26 triangles all together. The ones containing three shaded triangles have three units on a side. Thus the probability of choosing such a triangle is $\frac{3}{26}$. **Answer:** $\frac{3}{26}$

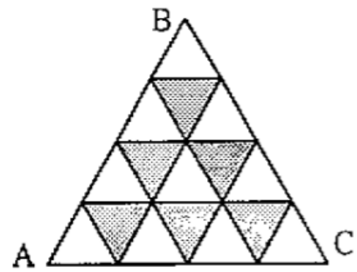
New York City Interscholastic Mathematics League

Solutions

F98J13. The given information yields two equations, one for the total number of coins and one for the total value of the coins (in cents): $\begin{cases} d + q = 20 \\ 10d + 25q = 380 \end{cases}$ → d= 8 and q =12. **Answer:** (8,12)

F98J14. To simplify the problem, change the number of subdivisions to a more manageable number. The first diagram on the right shows three subdivisions on a side. The result is 9 triangles. The second diagram shows four subdivisions on a side giving 16 triangles. For n subdivisions on a side, there will be n^2 disjoint triangles formed. **Answer:** 10,000

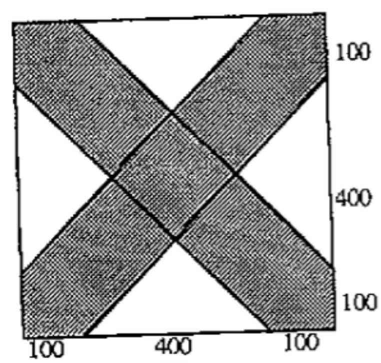




F98J15. The area of equilateral $\Delta ABC = \frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$. The area of equilateral $\Delta DBE = \frac{2^2\sqrt{3}}{4} = \sqrt{3}$. The area of the quadrilateral is the difference of these two areas. **Answer:** $8\sqrt{3}$

F98J16. Let x = Jack's average speed coming home and x + 20= his average speed going to work. The average speed on a round trip, following the same route is the harmonic mean of the two speeds. This gives the equation $\frac{2x(x+20)}{2x+20} = 37.5$ → $\frac{2x^2+40x}{x+10} = 75$ → $2x^2 + 40x = 75x + 750$ → $(2x + 25)(x-30) = 0$. Thus x=30mph. **Answer:** 30

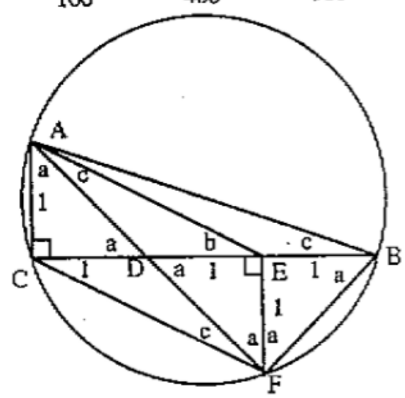
F98J17. There are several approaches to this problem. The easiest approach is to do the *reverse*. Instead of finding the shaded area, find the non- shaded area. Then subtract this result from the area of the square. The area of the square is $600^2 = 360{,}000$. The non-shaded region consists of four isosceles right triangles. Since the hypotenuse of each of these triangles is 400, each sides has length $200\sqrt{2}$. Thus the area of each triangle is $\frac{1}{2}\cdot(200\sqrt{2})\cdot(200\sqrt{2}) = 40{,}000$. The total non-shaded area is therefore 160,000, Subtracting this from the area of the square gives 200,000 square units, the shaded area. **Answer:** 200,000



F98J18. There are *several* solutions to this problem. A non- trigonometric solution is to:

- Extend $\overline{AD}$ its own length to point F. This means that ACFE is a parallelogram since its diagonals bisect each other.
- Connect $\overline{FB}$. This creates right ΔFEB ≅ ΔACD. Note that ∠AFB is a right angle.
- Circumscribe a circle about ΔABC. The circle will also pass through F since ∠AFB is a right angle which must be inscribed in a semi-circle with $\overline{AB}$ as diameter.
- Note that m∠CFA = c because it intercepts the same arc as ∠ABC. Since AEFC is a parallelogram, m∠FAE must = c. m∠DFE = a. Thus in ΔAFE, the sum of the three angle measures is 180: a + (b+90) + c =180 → a+b+c = 90

Answer: 90



January 24, 1999

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1998 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Senior A	F98S5	67
	F98S9	It should have read "Compute all real x"
Junior	F98J12 was eliminated	It should have read "exactly 3 shaded small triangles" The resulting triangles includes the original triangle. Answer: 4/27
	F98J14	2500

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML