NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE**

PART I: TIME 10 MINUTES **SPRING 1998**

S98B1 If $2^x \cdot 4^{2x} \cdot 8^{3x} = 512$, compute the value of x.

S98B2 Find the 1998th digit in the decimal expansion of $\frac{1}{7}$.

PART II: TIME 10 MINUTES **SPRING 1998**

S98B3 If $\sin x + \cos x = \frac{1}{4}$, find the numerical value of sin2x.

S98B4 Find the smallest positive integer greater than 2 which leaves a remainder of 2 when divided by 3, 4, 5, 6, 7, or 8.

PART III: TIME 10 MINUTES **SPRING 1998**

S98B5 If $3x + 4y = 7$ is perpendicular to $5x + ky = 11$, compute the value of k.

S98B6 The sides of a triangle measure 7, 10, and 13. Compute the length of the altitude to the side which has length 10.

ANSWERS

1. $\frac{9}{14}$
2. 7
3. $-\frac{15}{16}$
4. 842
5. $-\frac{15}{4}$
6. $4\sqrt{3}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO**

PART I: TIME 10 MINUTES **SPRING 1998**

S98B7 If $a = \log_4 32$ and $b = \log_8 16$, express the product ab in simplest form.

S98B8 In a circle, a chord of length 10 inches is 12 inches from the center. How far from the center is a chord of length 12 inches?

PART II: TIME 10 MINUTES **SPRING 1998**

S98B9 The interior angles of a regular polygon each measure 168°. How many sides does the polygon have?

S98B10 Find the sum of all distinct 4 digit numbers that can be formed using the digits 1, 3, 7, and 9 without repetition.

PART III: TIME 10 MINUTES **SPRING 1998**

S98B11 Find the number of points of intersection of the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 16$.

S98B12 Compute the numerical value of $\sin\left(Arc\sin\frac{3}{5} + Arc\cos\frac{5}{13}\right)$.

ANSWERS

7. $\frac{10}{3}$	9. 30	11. 0
8. $\sqrt{133}$	10. 133,320	12. $\frac{63}{65}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE**

PART I: TIME 10 MINUTES **SPRING 1998**

S98B13 If 5A = 7B and 9C = 11B, express the ratio C:A in simplest form.

S98B14 Find the area of the region bounded by the closed curve whose equation is $x^2 + 6x + y^2 + 8y = 0$.

PART II: TIME 10 MINUTES **SPRING 1998**

S98B15 A right triangle has area 48 and the legs are in the ratio 2:3. Find the length of the hypotenuse.

S98B16 Compute the value of $\log_{32}\sqrt[3]{0.25}$.

PART III: TIME 10 MINUTES **SPRING 1998**

S98B17 Compute the length of the diagonal of an isosceles trapezoid with sides 7, 8, 8 and 15.

S98B18 A student takes a 5 question multiple choice test with four choices on each question. He makes a random guess on each question. Compute the probability he gets at least 60% on the test.

ANSWERS

13. $\frac{55}{63}$
14. 25π
15. $\sqrt{208}$ or $4\sqrt{13}$
16. $-\frac{2}{15}$
17. 13
18. $\frac{53}{512}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR**

PART I: TIME 10 MINUTES **SPRING 1998**

S98B19 Write in simplest form: $\log 7 \div \log\frac{1}{7}$.

S98B20 Find the sum of the infinite series $6-4+\frac{8}{3}-\frac{16}{9}+\frac{32}{27}-\ldots$

PART II: TIME 10 MINUTES **SPRING 1998**

S98B21 A square and an equilateral triangle have equal perimeter. If the area of the triangle is $9\sqrt{3}$, find the area of the circle which is circumscribed around the square.

S98B22 Solve for all x, $0^\circ \le x \le 360^\circ$: $\sin 75^\circ - \cos 75^\circ = \sin x$

PART III: TIME 10 MINUTES **SPRING 1998**

S98B23 A sock drawer contains 5 white and 6 black socks. If 2 socks are drawn at random, find the probability that they match.

S98B24 The number a,b12,345 is divisible by 11. How many distinct ordered pairs (a,b) are possible?

ANSWERS

19. -1	21. $\frac{81\pi}{8}$	23. $\frac{5}{11}$
20. $\frac{18}{5}$	22. $45^\circ, 135^\circ$	24. 8

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FIVE**

PART I: TIME 10 MINUTES **SPRING 1998**

S98B25 Find the maximum value of $5\sin 4x\cos 4x$.

S98B26 Find the number of positive integral divisors of $(20)^5$.

PART II: TIME 10 MINUTES **SPRING 1998**

S98B27 Solve for all values of x: $3^{2x} - 10(3^x) + 9 = 0$.

S98B28 Find the units digit of the number 1! + 2! + 3! + 4! + ... + 1998!

PART III: TIME 10 MINUTES **SPRING 1998**

S98B29 If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, express $\log_{10} 150$ in simplest form in terms of a and b with no logarithms.

S98B30 In triangle ABC, AB = 9, AC = 5 and median AM = 4. Find the length of BC.

ANSWERS

25. $\frac{5}{2}$

26. 66

27. 0, 2

28. 3

29. b + 2 - a

30. $2\sqrt{37}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS SPRING, 1998 CONTEST ONE

S98B1 $2^x \cdot (2^2)^{2x} \cdot (2^3)^{3x} = 2^9$, $2^x \cdot 2^{4x} \cdot 2^{9x} = 2^9$. Therefore, $14x = 9$ or $x = \frac{9}{14}$.

S98B2 The decimal is $.\overline{142857}$. Since 1998 is a multiple of 6, the 1998th digit is the same as the 6th, which is 7.

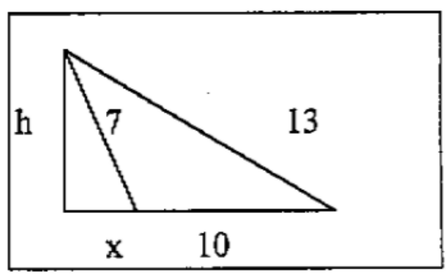
S98B3 $\sin x + \cos x = \frac{1}{4}$. $(\sin x + \cos x)^2 = \frac{1}{16}$.

$$\sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{1}{16}. \quad \sin 2x = 2\sin x \cos x = -\frac{15}{16}.$$

S98B4 The number is 2 more than the least common multiple of 3, 4, 5, 6, 7, and 8 or $2^3 \cdot 3 \cdot 5 \cdot 7 = 840$. $N = 842$.

S98B5 The slope of the first line is $-\frac{3}{4}$ and of the second is $-\frac{5}{k}$. Since they are negative reciprocals, $-\frac{3}{4} = \frac{k}{5}$. $4k = -15$ or $k = -\frac{15}{4}$.

S98B6 Since $13^2 > 10^2 + 7^2$, the triangle is obtuse.
$x^2 + h^2 = 49$
$(10 + x)^2 + h^2 = 169$.
Subtracting, $20x + 100 = 120$, $x = 1$.
$h = 4\sqrt{3}$.



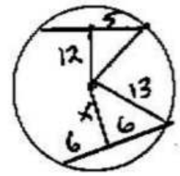
Note: This can also be solved using Heron's Formula.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS SPRING, 1998 CONTEST TWO

S98B7 $4^a = 32$ *or* $a = \frac{5}{2}$ and $8^b = 16$ *or* $b = \frac{4}{3}$. Thus, $ab = \frac{5}{2} \cdot \frac{4}{3} = \frac{10}{3}$

S98B8 $5^2 + 12^2 = r^2$ or r = 13.
$x^2 + 6^2 = 13^2$ or $x = \sqrt{133}$.



S98B9 The exterior angle is $12^\circ = \frac{360^\circ}{N}$ or N = 30. Alternately, an interior angle =

$$\frac{(N-2)180}{N} = 168.$$ 168N = 180N - 360 or N = 30.

S98B10 There are 4! such numbers. By symmetry, there are 6 of each number in each place. 6(1 + 3 + 7 + 9) = 120.
$120 \times 1 + 120 \times 10 + 120 \times 100 + 120 \times 1000 = 133{,}320.$

S98B11 The first is a circle with center (0,0) and radius 3. The second is a hyperbola with center (0,0) and x-intercepts (4,0) and (-4,0). There are no points in common. Or, adding both equations we obtain, $2x^2 = 25$ or $x^2 = \frac{25}{2}$. But then $y^2 = -\frac{7}{2}$ which is impossible.

S98B12 Using the sum formula, $\sin\left(Arc\sin\frac{3}{5} + Arc\cos\frac{5}{13}\right)$

$$= \sin\left(Arc\sin\frac{3}{5}\right)\cos\left(Arc\cos\frac{5}{13}\right) + \cos\left(Arc\sin\frac{3}{5}\right)\sin\left(Arc\cos\frac{5}{13}\right)$$

$$= \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{63}{65}.$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS SPRING, 1998 CONTEST THREE

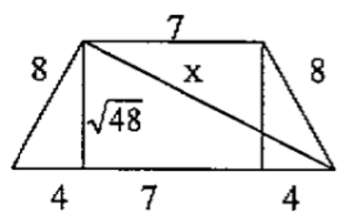
S98B13 C:A = $\dfrac{\frac{11B}{9}}{\frac{7B}{5}} = \dfrac{55}{63}$.

S98B14 Completing the square, $(x+3)^2 + (y+4)^2 = 25$. This is a circle with radius 5. Therefore, the area is 25π.

S98B15 $\frac{1}{2} \cdot 2x \cdot 3x = 48$ or x = 4. The legs are 8 and 12 and the hypotenuse is $\sqrt{208}$.

S98B16 $32^x = 2^{\frac{-2}{3}}$ or $2^{5x} = 2^{\frac{-2}{3}}$ or $x = -\frac{2}{15}$.

S98B17 $\left(\sqrt{48}\right)^2 + 11^2 = x^2$ or $x = 13$.



S98B18 P(3 correct) = ${}_5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{1024}$

P(4 correct) = ${}_5C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 = \frac{15}{1024}$

P(5 correct) = $\left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

P(3, 4 or 5) = $\frac{106}{1024} = \frac{53}{512}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS SPRING, 1998 CONTEST FOUR

S98B19 $\log\frac{1}{7} = -\log 7$. $\frac{\log 7}{-\log 7} = -1$

S98B20 This is an infinite geometric progression with common ratio $-\frac{2}{3}$. Thus, the sum is $s = \frac{a}{1-r} = \frac{6}{1-\left(-\frac{2}{3}\right)} = \frac{6}{\frac{5}{3}} = \frac{18}{5}$.

S98B21 The side of the triangle is 6. Its perimeter is 18. The side of the square is then $\frac{9}{2}$. The radius of the circumscribed circle is $\frac{s}{2}\sqrt{2} = \frac{9}{4}\sqrt{2}$. Its area is $\pi\left(\frac{9}{4}\sqrt{2}\right)^2 = \frac{81}{8}\pi$.

S98B22 Using $\sin(45^\circ + 30^\circ)$ - $\cos(45^\circ + 30^\circ)$ and remembering that
$\sin 45^\circ = \cos 45^\circ$,
$\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ - (\cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ) =$
$2\sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}$. $\sin x = \frac{\sqrt{2}}{2}$, $x = 45^\circ, 135^\circ$.

S98B23 P(2 whites) = $\frac{5}{11} \cdot \frac{4}{10} = \frac{20}{110}$; P(2 blacks) = $\frac{6}{11} \cdot \frac{5}{10} = \frac{30}{110}$;
P(match) = $\frac{20}{110} + \frac{30}{110} = \frac{50}{110} = \frac{5}{11}$.

S98B24 The odd placed integers must differ from the even placed integers by either 0 or a multiple of 11. Either 9 + a = 6 + b or 9 + a = 6 + b + 11. The first gives b - a = 3 giving 6 six solutions, (1,4), (2,5), (3,6), (4,7), (5,8), (6,9). The second gives a - b = 8 or (8,0) and (9,1), a total of 8 possible ordered pairs.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS SPRING, 1998 CONTEST FIVE

S98B25 $\sin 4x \cos 4x = \dfrac{\sin 8x}{2}$. Since the maximum value of the sine function is 1, the maximum value of $\dfrac{5}{2}\sin 8x$ is $\dfrac{5}{2}$.

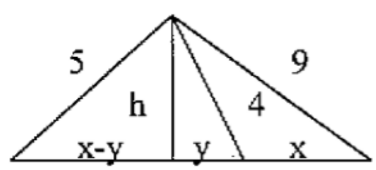
S98B26 $20 = 2^2 \cdot 5$. $20^5 = 2^{10} \cdot 5^5$. The number of integral factors is the product of 1 more than each exponent. $F = (10 + 1)(5 + 1) = 66$.

S98B27 Let $3^x = y$. Therefore, $y^2 - 10y + 9 = 0$ or $y = 9$, $y = 1$. If $y = 9$, $x = 2$ and if $y = 1$, $x = 0$.

S98B28 Since all the numbers past 5! have a 0 units digit, the digit will be the units digit of 1 + 2 + 6 + 24 which is 3.

S98B29 $\log_{10} 150 = \log_{10} \dfrac{300}{2} = \log_{10} \dfrac{3 \cdot 100}{2} = \log_{10} 3 + \log_{10} 100 - \log_{10} 2$ $= b + 2 - a$.

S98B30 Draw altitude AD.



(1) $h^2 + (x - y)^2 = 25$
(2) $h^2 + (x + y)^2 = 81$
(3) $h^2 + y^2 = 16$.
Adding (1) and (2), $2h^2 + 2x^2 + 2y^2 = 106$
$h^2 + x^2 + y^2 = 53$
Subtracting (3), $x^2 = 37$ or $x = \sqrt{37}$. $BC = 2x = 2\sqrt{37}$.