NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE**

PART I: TIME 10 MINUTES **FALL 1996**

F96B1 What two digit number is exactly five times the sum of its digits?

F96B2 How many ordered pairs of integers (x,y) satisfy $2^4 + x^y = 2^5$?

PART II: TIME 10 MINUTES **FALL 1996**

F96B3 Find the sum of the first 3 coefficients of the expansion of $(x - 1)^7$.

F96B4 A circle is inscribed in a triangle with sides of length 9, 12, and 14. The side of length 12 is divided into two segments by the point of tangency. Find the ratio a:b of the shorter segment to the longer segment where a and b are relatively prime positive integers.

PART III: TIME 10 MINUTES **FALL 1996**

F96B5 If $720a = b^3$, and a and b are positive integers, find the smallest possible a.

F96B6 Find the sum of the infinite series $\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}+\ldots$.

ANSWERS

1. 45	3. 15	5. 300
2. 5	4. 7:17	6. 4/9

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO**

PART I: TIME 10 MINUTES **FALL 1996**

F96B7 The first three terms of an arithmetic progression are given by $x - 3$, $x + 5$, and $3x + 3$. Compute the value of the fourth term.

F96B8 A median of length 3 is drawn to a side of a triangle of length 8, and divides the triangle into 2 triangles with equal perimeters. Find the area of the original triangle.

PART II: TIME 10 MINUTES **FALL 1996**

F96B9 If $\frac{x-2}{x+2}$ and $\frac{x+2}{x-2}$ are both integers, find x.

F96B10 Seven lines are drawn in a plane. No two are parallel and no three are concurrent. Into how many regions is the plane divided?

PART III: TIME 10 MINUTES **FALL 1996**

F96B11 How many ordered pairs of positive integers satisfy $3x + 2y = 100$?

F96B12 Between 6 o'clock and 7 o'clock, the hands of the clock twice form an angle of 110^0. How many minutes apart are these two times?

ANSWERS

7. 26	9. 0	11. 16
8. 12	10. 29	12. 40

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE**

PART I: TIME 10 MINUTES **FALL 1996**

F96B13 The vertex of the parabola whose equation is $y = x^2 + kx + 16$ is on the x axis. Find all possible values of x.

F96B14 If $3^x = 9^{y-1}$ and $4^x = 8^{y+1}$, find the ratio x:y where x and y are relatively prime positive integers.

PART II: TIME 10 MINUTES **FALL 1996**

F96B15 If $10^{\log_{10} 7} = 3x + 2$, compute the value of x.

F96B16 In a 10 mile race, Bill beats Bob by 2 miles, and Bill beats Ross by 4 miles. If they continue their same speeds, by how many miles does Bob beat Ross?

PART III: TIME 10 MINUTES **FALL 1996**

F96B17 The number x has exactly 15 positive integral factors. Two of them are 27 and 15. Find the value of x.

F96B18 A diagonal is drawn in a rectangle with dimensions 8 x 15. A circle is inscribed in each of the two triangles formed. Find the distance between the centers of the circles.

ANSWERS

13. +8, -8	15. $\frac{5}{3}$	17. 2025
14. 12:7	16. $2\frac{1}{2}$	18. $\sqrt{85}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR**

PART I: TIME 10 MINUTES **FALL 1996**

F96B19 If $x + \frac{1}{x} = 3$, compute the value of $x^4 + \frac{1}{x^4}$.

F96B20 If [x] represents the greatest integer less than or equal to x, compute the value of x for which x[x] = 40.

PART II: TIME 10 MINUTES **FALL 1996**

F96B21 Compute the length of the diagonal of an isosceles trapezoid with sides 6, 6, 10, and 18.

F96B22 If the prime factorization of N! is $2^{34} \cdot 3^{17} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31$, find N.

PART III: TIME 10 MINUTES **FALL 1996**

F96B23 A regular 12 sided polygon is inscribed in a circle with radius r. Express the area of the polygon in simplest form in terms of r.

F96B24 Find all ordered pairs of positive integers (x,y) which satisfy $xy + 4x - 13 = x^2$.

<u>**ANSWERS**</u>

19. 47	21. $6\sqrt{6}$	23. $3r^2$
20. 20/3	22. 36	24. (1,10), (13,10)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FIVE**

PART I: TIME 10 MINUTES **FALL 1996**

F96B25 What is the measure of the obtuse angle formed by the hour and minute hands of a clock at 2:45?

F96B26 There are eight different flavored donuts on a table when Tubby takes at least two but at most six. How many different combinations can he take?

PART II: TIME 10 MINUTES **FALL 1996**

F96B27 In a sequence of numbers each number is the sum of the two immediately preceding it; that is, $S_n = S_{n-2} + S_{n-1}$. If the eleventh number is 343 and the twelfth number is 555, find the first number.

F96B28 Find the sum of the digits of the first 100 positive integers which are multiples of 3.

PART III: TIME 10 MINUTES **FALL 1996**

F96B29 In a polygon of N sides, 44 diagonals can be drawn. Find N.

F96B30 Dwight and Adlai take turns spinning a spinner with seven equally likely results, the numbers 1, 2, 3, 4, 5, 6, or 7. The winner is the person who spins a 6 or a 7 first. If Dwight spins first, compute his probability of winning.

ANSWERS

25. $172\frac{1}{2}$	27. 2	29. 11
26. 238	28. 1002	30. 7/12

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

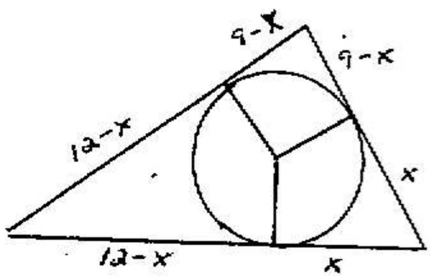
SENIOR B SOLUTIONS FALL, 1996 CONTEST ONE

F96B1 $10t + u = 5(t + u)$; $5t = 4u$; $t = 4$ and $u = 5$. Therefore, the two digit number is 45.

F96B2 $x^y = 16$. Thus, (16,1), (4,2), (-4,2), (2,4), and (-2,4) are all solutions. There are five ordered pairs.

F96B3 Using Pascal's triangle, the first three coefficients are 1, -7, and 21. The sum is 15.

F96B4 $12 - x + 9 - x = 14$
$2x = 7$
$x = 7/2$
Ratio $= (7/2)/(17/2) = 7/17$



F96B5 The exponents must be multiples of 3. $2^4 \cdot 3^2 \cdot 5 \cdot a = b^3$.
Therefore,
$a = 2^2 \cdot 3 \cdot 5^2 = 300$.

F96B6 Setting this up as the sum of an infinite number of infinite geometric series,

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\ldots$$
$$\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\frac{1}{4^5}+\ldots$$
$$\frac{1}{4^3}+\frac{1}{4^4}+\frac{1}{4^5}+\frac{1}{4^6}+\ldots$$
$$\ldots$$

we obtain $S = \frac{1}{3}+\frac{1}{12}+\frac{1}{48}+\ldots=\frac{4}{9}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1996 CONTEST TWO

F96B7 x + 5 - (x - 3) = 3x + 3 - (x + 5); 8 = 2x - 2; x = 5
The terms are 2, 10, 18, and 26.

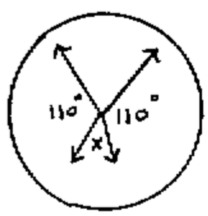
F96B8 If median AM divides triangle ABC into two triangles of equal perimeter then triangle ABC is isosceles. Then AM is also an altitude and $A = \frac{1}{2} \cdot 3 \cdot 8 = 12$.

F96B9 The only integers whose reciprocals are integers are 1 and -1. Since $\frac{x-2}{x+2}$ cannot be 1, $\frac{x-2}{x+2} = -1$. x - 2 = -(x + 2);
x - 2 = -x - 2; x = 0.

F96B10 Starting with one line, two lines, three lines, etc. it can easily be shown that the number of regions is one more than the sum of the numbers of the lines. With seven lines, the number of regions is (1 + 2 + 3 + 4 + 5 + 6 + 7) + 1 = 29.

F96B11 $y = \frac{100-3x}{2}$, x can be any even integer from 2 to 32. There are 16 pairs.

F96B12 If the hour hand moves x°, the minute hand moves 12x°. But from the diagram, the minute hand also moves (220 + x)°. Thus, 220 + x = 12x or x = 20 and the minute hand moves 240° which is 40 minutes.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1996 CONTEST THREE

F96B13 This will occur when $x^2 + kx + 16$ is a perfect square. $k = \pm 8$.

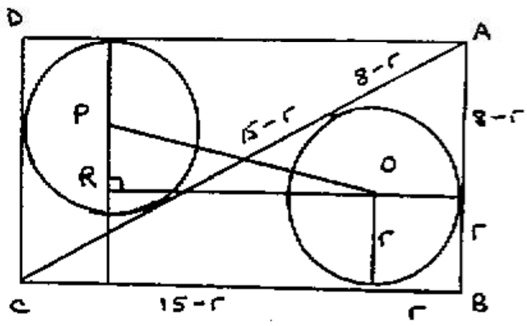
F96B14 $3^x = (3^2)^{y-1}$ $x = 2y - 2$
$(2)^{2x} = 2^{3(y+1)}$ $2x = 3y + 3$
Therefore, $x = 12$, $y = 7$. $x:y = 12:7$

F96B15 $10^{\log_{10} 7} = 7.$ $3x + 2 = 7$. $x = 5/3$

F96B16 Since Bob has completed 8 miles and Ross 6, when Bob completes another 2 miles, $\frac{8}{6} = \frac{2}{x}$, $8x = 12$, $x = 1.5$, Ross completes another 1.5 miles. Bob beats Ross by $10 - 7.5 = 2.5$.

F96B17 The number must be $p_1{}^4p_2{}^2$ where p_1 and p_2 are primes, since the number of factors is $(4 + 1)(2 + 1) = 15$. Since $27 = 3^3$, and $15 = 3^1 \cdot 5^1$, the number must be $3^4 \cdot 5^2 = 2025$.

F96B18 $AC = 17 = 23 - 2r$. $r = 3$.
$OR = 15 - 2r = 9$
$RP = 8 - 2r = 2$
$OP = \sqrt{85}$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

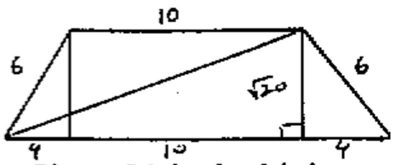
SENIOR B SOLUTIONS FALL, 1996 CONTEST FOUR

F96B19 $\left(x+\frac{1}{x}\right)^2=9$. $x^2+2+\frac{1}{x^2}=9$. $x^2+\frac{1}{x^2}=7$.

$\left(x^2+\frac{1}{x^2}\right)^2=49$. $x^4+2+\frac{1}{x^4}=49$. $x^4+\frac{1}{x^4}=47$.

F96B20 The number is obviously between 6 and 7. $[x] = 6$. $6x = 40$ which yields $x=\frac{40}{6}=\frac{20}{3}$.

F96B21 $\left(\sqrt{20}\right)^2+14^2=x^2$. $x=\sqrt{216}=6\sqrt{6}$



F96B22 Since 31 is the highest prime, N must be between 31 and 36. Using the factors of 3 (2 can also be used), 36! has a factor of 3^{17}. That is, one for the 12 multiples of 3, one extra for the four multiples of 9, and a seventeenth for 27. Therefore, N = 36.

F96B23 The polygon contains 12 congruent isosceles triangles with legs r and vertex angle 30°. $A = 12\cdot\frac{1}{2}\cdot r\cdot r\cdot\sin 30° = 3r^2$.

F96B24 $xy + 4x - 13 = x^2$, $xy + 4x - x^2 = 13$. $x(y + 4 - x) = 13$. Since 13 is prime and x is a positive integer, x = 1 or x = 13. (1,10) and (13,10).

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B SOLUTIONS FALL, 1996 CONTEST FIVE

F96B25 The minute hand is at 270°. The hour hand is at $60° + \frac{3}{4}\cdot 30° =$

$82\frac{1}{2}$. $90 + 82\frac{1}{2} = 172\frac{1}{2}°$.

F96B26 The number of subsets of 8 objects is $2^8 = 256$. Subtracting 1 subset with 0 objects, 8 with 1 object, 8 with 7 objects and 1 with 8 objects, the number is 238.

F96B27 Going backwards, $a_{10} = 212$, $a_9 = 131$, $a_8 = 81$, $a_7 = 50$, $a_6 = 31$, $a_5 = 19$, $a_4 = 12$, $a_3 = 7$, $a_2 = 5$, and $a_1 = 2$.

F96B28 By symmetry, the units digit will have 10 of each digit. $10(1 + 2 + 3 + \ldots) = 450$. Likewise, the tens digit will total 450. The hundreds digit will contain 33 zeros, 33 ones, 33 twos, and one 3. $450 + 450 + 33 + 66 + 3 = 1002$.

F96B29 The number of diagonals is $n(n - 3)/2$. Therefore, $n^2 - 3n - 88 = 0$. $n = 11$.

F96B30 His probability of winning is $\frac{2}{7}+\frac{5}{7}\cdot\frac{5}{7}\cdot\frac{2}{7}+\frac{5}{7}\cdot\frac{5}{7}\cdot\frac{5}{7}\cdot\frac{5}{7}\cdot\frac{2}{7}+\ldots$. This is an infinite geometric progression whose sum is $\dfrac{\frac{2}{7}}{1-\frac{25}{49}} = \frac{14}{24} = \frac{7}{12}$.

January 20, 1997

Dear Math Team Coach,

Enclosed is your copy of the ~~Spring~~ Fall, 1996 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Senior B	F96B13 was eliminated	It should have read "Find all possible values of k." Then the given answers of +8, -8 were correct.

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML