NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE

PART I TIME: 10 MINUTES FALL 1996

F96S1 Solve for all real values of x: $||5x| + 12| < 13$.

F96S2 At the county fair, there is a game of chance that requires tossing a flat circular disk 2" in diameter onto a surface that has a rectangular grid painted on the surface. The grid forms squares that measure 2.5" on a side. The player gets 2 disks to toss for $1. If one of the disks lands entirely within a square, the game is over and the player wins the kewpie doll. What is the probability of winning the doll? (If the disk fails to land on the surface entirely, the player gets another toss for free.)

PART II TIME: 10 MINUTES FALL 1996

F96S3 For any integer x, let f(x) be the remainder when the sum of the digits of $x^2 + x$ is divided by 3. Compute f(1996).

F96S4 Compute the smallest two digit prime divisor of $12! + 1$.

PART III TIME: 10 MINUTES FALL 1996

F96S5 The product of three consecutive positive even integers is a seven digit number whose first and last digits are 8. Compute the second of the three integers.

F96S6 The vertices of $\triangle ABC$ are as follows: A is (-5,10), B is the reflection of A on the line $-2x + y = 5$, and C is the reflection of B on the line $x + 2y = 25$. Compute the length of the median from B to AC.

ANSWERS

F96S1 $-1/5 < x < 1/5$

F96S2 49/625

F96S3 2

F96S4 13

F96S5 204

F96S6 $\sqrt{65}$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO

PART I TIME: 10 MINUTES FALL 1996

F96S7 Let O be the center of a circle tangent to side $\overline{BC}$ and to the extensions of sides $\overline{AB}$ and $\overline{AC}$ outside of $\triangle ABC$. Let F be the point of tangency on $\overline{AB}$ extended. If the perimeter of $\triangle ABC$ is 1996, compute AF.

F96S8 If $x + y = a$, $x^2 + y^2 = b$, and $x^3 + y^3 = c$, write an equation in simplest form expressing b explicitly in terms of both a and c.

PART II TIME: 10 MINUTES FALL 1996

F96S9 Compute the last digit of $(19^{96} + 96^{19})^{1996}$

F96S10 Compute all the real values of x that satisfy $25^x + \dfrac{15,625}{625^x} = \dfrac{15,626}{5^x}$.

PART III TIME: 10 MINUTES FALL 1996

F96S11 The sides of a triangle are 1800, 1250 and 942. Let O be the center of the circle inscribed in the triangle. Compute the ratio of the area of the triangle to the radius of the inscribed circle.

F96S12 Let $\sqrt{2} = 1.4$ and $a_n = [n\sqrt{2}]$ be the n^{th} term of the sequence $\{a_n\}$ for $n \geq 1$ (where [x] is equal to the greatest integer less than or equal to x) Let b_n be the n^{th} term of the sequence $\{b_n\}$ of the positive integers, in order of increasing size, consisting of all positive integers not appearing in $\{a_n\}$. Compute b_{100}.

ANSWERS
F96S7 998
F96S8 $b = (a^3 + 2c)/3a$
F96S9 1
F96S10 $x = 0$ or $x = 2$
F96S11 1996
F96S12 349

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE

PART I TIME: 10 MINUTES FALL 1996

F96S13 Compute the sum of all positive proper fractions whose denominators are no greater than 100.

F96S14 Three positive integers in an arithmetic progression are such that if the smallest is increased by three, the new triplet forms a geometric progression or if the largest is increased by six, the new triplet also forms a geometric progression. Write the three integers in increasing order.

PART II TIME: 10 MINUTES FALL 1996

F96S15 Compute the tangent of the angle between the internal diagonal of a cube and the diagonal of one of its faces.

F96S16 Let ABCD be a square with side of length 3. Let E, F, G and H be points on sides AB, BC, CD and DA, respectively, such that AE = BF = CG = DH = 1. By drawing AF, BG, CH, and DE an interior square is formed whose vertices are the points of intersection of these segments. Compute the area of the interior square formed.

PART III TIME: 10 MINUTES FALL 1996

F96S17 $(1+\sqrt{3})^{32} = 2^n(a+b\sqrt{3})$ where a and b are relatively prime integers. Compute the integer value of n.

F96S18 Let x be the smallest positive integer such that $N = 2592x$ is both a perfect cube and a perfect fifth power. Compute $[\log_6 x]$ where $[a]$ is the greatest integer less than or equal to a.

ANSWERS
F96S13 2,475
F96S14 24, 36 and 48
F96S15 $\sqrt{2}/2$
F96S16 3.6
F96S17 16
F96S18 10

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
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PART I   TIME: 10 MINUTES   FALL 1996

F96S19

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
...

In the triangular array of integers above, each integer in successive rows is the sum of the integers in the preceeding row immediately above it and to the left and right (assuming 0 in positions where integers do not appear). For example, the 7 in row 4 is the sum of 2,3 and 2 in row 3. If the sum of the integers appearing in the center positions of the first 20 rows is even, write "EVEN"; If not, write "ODD".

F96S20 For positive integer x, let d(x) be the single digit arrived at when the digits of x are added and, if necessary, the digits of the sum are added and so on. Compute $d(31^2) + d(31^4) - d(31^8) - d(31^{16})$.

PART II   TIME: 10 MINUTES   FALL 1996

F96S21 Write a four digit perfect square whose first two digits are identical and whose last two digits are identical.

F96S22 Write all the ordered triplets of positive integers (a,b,c) that satisfy $a < b < c$, $a(b+c)=27$, $b(a+c)=32$ and $c(a+b)=35$ simultaneously.

PART III   TIME: 10 MINUTES   FALL 1996

F96S23 Finding herself without a calculator, Miss Candomath approximates 2^{10} as 10^3 and computes the base ten logarithms of 2, 3, 4, 5, and 6 to the nearest tenth. Write these decimal values in order from smallest to largest.

F96S24 If $|y| < 1000$, compute the largest value of y such that $x^3 - 36x + y = 0$ for integer x.

ANSWERS

F96S19 EVEN
F96S20 0
F96S21 7744
F96S22 (3,4,5)
F96S23 .3, .5, .6, .7 and .8
F96S24 935

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE

PART I TIME: 10 MINUTES FALL 1996

F96S25 Compute the range of $y = \dfrac{1 - \cos\left(\dfrac{\pi(x+3)!}{2}\right)}{2}$, (where $n!$ represents "n factorial") if its domain is restricted to the nonnegative integers.

F96S26 Let C(n) be the integer given by the numeral formed by writing the consecutive integers from 1 to n in order starting with 1. Compute the power of 10 that appears when C(100) is written in scientific notation.

PART II TIME: 10 MINUTES FALL 1996

F96S27 Compute the smallest prime factor of $3^{65,536} - 2^{65,536}$

F96S28 If $x + \dfrac{1}{x} = 2\cos 15°$, compute $x^4 + \dfrac{1}{x^4}$.

PART III TIME: 10 MINUTES FALL 1996

F96S29 If p and q are the roots of $\sqrt{17}\,x^2 - \sqrt{51}\,x + \sqrt{3} = 0$, compute $\dfrac{\dfrac{p}{q} - \dfrac{q}{p}}{p - q}$

F96S30 Within a square, a circle is inscribed within which a square is inscribed within which a circle is inscribed within which a square is inscribed and so on indefinitely. Within every square the region exterior to its inscribed circle is shaded. If the side of the largest square is $\sqrt{2}$, compute the total area of the shaded regions.

ANSWERS
F96S25 {0,1}
F96S26 191
F96S27 5
F96S28 1
F96S29 $\sqrt{17}$
F96S30 $4 - \pi$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
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SOLUTIONS

F96S1 The given inequality is equivalent to $-13 < |5x| + 12 < 13$.
Thus, $-25 < |5x| < 1$ which is solvable only for $0 \le |5x| < 1$ which gives $-1/5 < x < 1/5$.

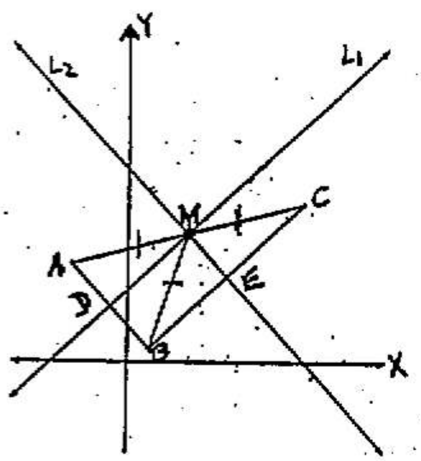
F96S2 Any square on the grid has an area of $(5/2)^2$. Within any square, the center of the disk must land within a square region, concentric with the grid square, of area nearly $(1/2)^2$ so that the edge of the square does not touch a line. The probability of this occurring is, therefore, $(1/2)^2/(5/2)^2$ or $1/25$. Winning the game can occur in two ways, being successful on the first toss or being successful only on the second toss. Therefore, the probability of winning is $1/25 + (24 \cdot 1)/625$. The probability of winning is, therefore, $49/625$.

F96S3 An integer and the sum of its digits have the same remainder when divided by 3. If $x/3$ has remainder 0 or 2, then $(x^2 + x)/3$ has remainder 0. If $x/3$ has remainder of 1, then $(x^2 + x)/3$ has remainder 2. 1996 has a remainder of 1 when divided by 3. Therefore, $f(1996) = 2$.

F96S4 $12! + 1 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 1$. Clearly, 11 is not a divisor of the sum. Checking 13, by replacing each integer with one that is congruent to itself mod 13, we have
$12! + 1 \equiv (-1)(-2)(-3)(-4)(-5)(-6)(6)(5)(4)(3)(2)(1) + 1 \equiv 6^2 5^2 4^2 3^2 2^2 + 1 \equiv (-3)(-1)(3)(-4)(4) + 1 \equiv -144 + 1 \equiv -143$ which is congruent to 0 mod 13. Therefore, the smallest two digit prime divisor is 13. (This is also readily given by Wilson's Theorem which states that $(p-1)! + 1$ is divisible by p for prime p.)

F96S5 Let the integers be represented by $n - 2$, n, and $n + 2$. Their product is $n^3 - 4n$ and this must be greater than 8,000,000. Therefore, n is between 200 and 300 (which is clearly much too large). Since the product ends in 8, the integers must have last digits of 2, 4 and 6, respectively. Since $210^3 - 840 > 9{,}000{,}000$, the integers must be 202, 204 and 206. n is, therefore, 204.

F96S6 Let L1 be the line $-2x + y = 5$ and L2 be the line $x + 2y = 25$. Clearly, L1 is perpendicular to L2 and their point of intersection, M, is (3,11). Let D be the point of intersection of AB and L1 and E be the point of intersection of BC and L2. A, M and C are easily seen to be collinear due to the perpendicular and parallel relationships of the lines. By the properties of line reflections, it is easily seen that $AM \cong MB \cong MC$ and, therefore, M is the midpoint of AC. The median BM is congruent to AM whose length, by the distance formula, is $\sqrt{65}$.



NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
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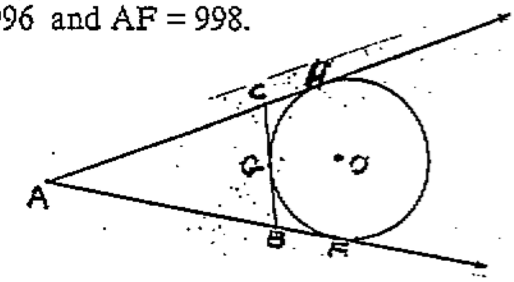
SOLUTIONS

F96S7 Let G be the point of tangency of circle O on BC.
AF = AB + BF = AB + BG and AF = AH = AC + CH = AC + CG.
Therefore, 2AF = AB + AC + BG + CG = AB + AC + BC = 1996 and AF = 998.
(Circle O is said to be an escribed circle of $\triangle$ABC.)

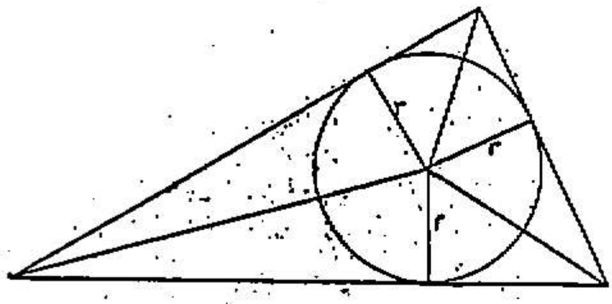


F96S8 $a^2 = (x + y)^2 = x^2 + y^2 + 2xy = b + 2xy$. Therefore, $xy = (½)(a^2 - b)$.
$c = x^3 + y^3 = (x + y)(x^2 + y^2 - xy) = a(b - (½)(a^2 - b)) = (½)a(3b - a^2)$ giving $2c = 3ab - a^3$ and $b = (a^3 + 2c)/3a$.

F96S9 The last digit of 19^n is 1 if n is even. The last digit of 96^n is 6 for all n>1. Therefore, the last digit of $19^{96} + 96^{19}$ is 7. The last digit of powers of integers ending in 7 follows the cycle 1,7,9,3,1,7,9,3,... Since 1996 is divisible by 4, the last digit of the number in question is 1.

F96S10 Rewriting the equation using powers of 5 and transposing yields the equation:
$5^{2x} - \frac{15,626}{5^x} + \frac{15,625}{5^{4x}} = 0$. Let $y = 5^x$ and multiplying all terms by y^4 yields the quadratic form:
$y^6 - 15,626y^3 + 15,625 = 0$. Factoring gives $(y^3 - 15,625)(y^3 - 1) = 0$ giving $y^3 = 1$ or $y^3 = 15,625$. Thus, y = 1 or y = 25 and, finally, x = 0 or x = 2.

F96S11 Drawing the radii of O of length r to the sides of the triangle, it is seen that the area of the triangle is equal to (½)rp, where p is the perimeter of the triangle. Therefore, area/r = (½)p = (½)(3992) = 1996.



F96S12 The sequence $\{n\sqrt{2}\} = \{1.4n\} = 1.4, 2.8, 4.2, 5.6, 7.0, ...$ and it is easily noticed that whenever the decimal part of the number is 8 or 6, the next number will have "skipped" an integer. The sequence of digits appearing in the tenth place is 4, 8, 2, 6, 0, 4, 8, 2, 6, 0, 4, 8, 2, 6, 0, ...
Therefore, $\{a_n\}$ will skip integers in positions 3, 6, 10, 13, 17, 20, ... which are the elements of $\{b_n\}$. This gives a two part generating formula for $n \geq 1$, $b_{2n-1} = 3 + 7(n - 1)$ and $b_{2n} = 6 + 7(n - 1)$.
$b_{100} = b_{2 \cdot 50} = 6 + 7(49) = 349.$

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F96S13

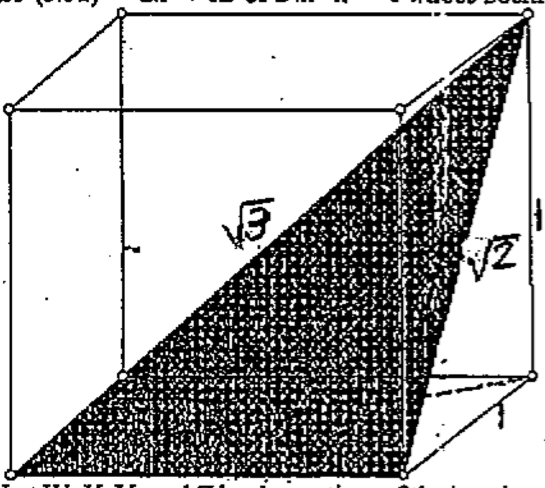
$$\frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \ldots + \left(\frac{1}{100} + \frac{2}{100} + \ldots + \frac{99}{100}\right)$$

$$= \sum_{k=2}^{100} \frac{(k-1)k}{2k}$$

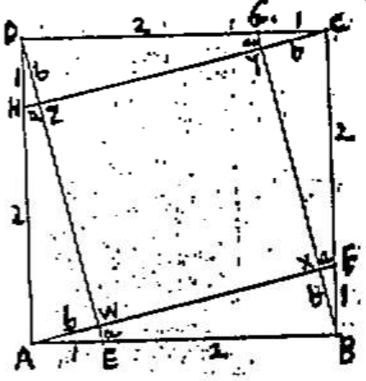
$$= \frac{1}{2}\sum_{k=1}^{99} k = \frac{1}{2}\,\frac{99 \cdot 100}{2} = 2{,}475$$

F96S14 Let x, y and z be the integers in increasing order. The given information leads to the three equations: $y = (x+z)/2$, $y^2 = (x+3)z$ and $y^2 = x(z+6)$. The latter two equations lead us to $z = 2x$. Using this in the first equation, we have $y = 3x/2$. Using these two results in the second equation gives the quadratic $(3x/2)^2 = 2x^2 + 6x$ or $24x - x^2 = 0$ whose positive root is $x = 24$. The three integers are, therefore, 24, 36 and 48.

F96S15 Without loss of generality, assume the edge of the cube has length 1. Therefore, the face diagonal has length $\sqrt{2}$. Using the Pythagorean Theorem for the right triangle formed by the internal diagonal, an edge and a face diagonal, the length of the internal diagonal is found to be $\sqrt{3}$. Using the law of cosines on the triangle and letting θ be the angle in question, we find $\cos\theta = \sqrt{6}/3$.



$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{9}{6} - 1} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$



F96S16 Let K be the area we seek. Let W, X, Y, and Z be the vertices of the interior square. Clearly, $\Delta ABF \cong \Delta BCG \cong \Delta CDH \cong \Delta DAE$ and $\Delta AEW \cong \Delta BFX \cong \Delta CGY \cong \Delta DHZ$. Therefore, $K = 9 - (4 \cdot \text{Area } \Delta ABF - 4 \cdot \text{Area } \Delta AEW)$. Let a be the length of EW and b be the length of $BX = AW$. Since $\Delta AEW \sim \Delta ABX$, $b = 3a$. We also have $a^2 + b^2 = 1$. Therefore, $10a^2 = 1$ or $a^2 = 1/10$ and $b^2 = 9/10$. Area $\Delta AEW = (1/2)ab = 3/20$. Thus, $K = 9 - (4(3/2) - 4(3/20)) = 72/20 = 3.6$.

F96S17 $(1+\sqrt{3})^{32} = (4+2\sqrt{3})^{16} = 2^{16}(2+\sqrt{3})^{16} = 2^{16}(7+4\sqrt{3})^8$
Clearly, the maximum n is 16, since 2 does not divide $7 + 4\sqrt{3}$.

F96S18 Repeated division of 2592 by 6 reveals that $N = 6^4 2x = 2^5 3^4 x$.
Let $x = 2^a 3^b$ and, therefore, $N = 2^{a+5}3^{b+4}$. The smallest positive N of this form that is both a perfect cube and a perfect fifth power is $2^{15}3^{15}$. Therefore, $a = 10$ and $b = 11$ and $x = 2^{10}3^{11} = 6^{10}3$.
$[\log_6 6^{10}3] = [10 + \log_6 3]$ and, since $0 < \log_6 3 < 1$, the answer is 10.

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F96S19 Since each row is symmetrical, the integer in the center position would be the sum of one of the following three possibilities:

	EVEN EVEN EVEN	ODD EVEN ODD	ODD ODD ODD	EVEN ODD EVEN
yielding, respectvely,	EVEN	EVEN	ODD	ODD

Since the array starts with an ODD in the center position, the first two possibilities cannot occur. Therefore, all integers in the center position are ODD. The sum of twenty odd integers is an <u>even</u> integer.

F96S20 For any integer x that is not a multiple of 9, d(x) is the remainder when x is divided by 9. (If x is a multiple of 9, d(x) = 9.) If a and b have remainders upon division by 9 of m and n, respectively, then d(ab)=d(mn). Therefore, $d(x^n)=d(d(x)^n)$. Since d(31)=4, we have
$d(31^2) = d(16) = 7$, $d(31^4) = d((31^2)^2) = d(d(31^2)^2) = d(49) = 4$, $d(31^8) = d((31^4)^2) = d(d(31^4)^2) = d(16) = 7$, and, finally, $d(31^{16}) = d((31^8)^2) = d(d(31^8)^2) = d(49) = 4$. Therefore, the sum we seek is 4 + 7 - 4 - 7 = 0.

F96S21 Let the two distinct digits be a and b and the number be $a \cdot 10^3 + a \cdot 10^2 + b \cdot 10 + b$ or 11(100a + b). Since the number is a perfect square, 100a + b must be divisble by 11. The usual divisbility rule for 11 requires that a+b be divisible by 11. Therefore, the possible numbers are 2299, 3388, 4477, 5566, 6655, 7744, 8833, and 9922. Also, since perfect squares have specific last digits (0,1,4,5,6 and 9 with 5 appearing as 25), the field is narrowed to 2299, 5566, and 7744. The number must also be divisible by 121, we can examine the cofactors of 121. They are, respectively: 19, 46 and 64. Therefore, the number we seek is unique and it is 7744.

F96S22 Since a, b and c are integers, each of the three equations along with the inequality leads to a restricted set of possibilities. Namely, from the first equation, a = 1 and b+c = 27 or a = 3 and b+c=9; from the second equation, b = 2 and a+c = 16 or b = 4 and a+c=8; from the third equation, c = 5 and a+b=7. Since the third equation is only solvable in one way, we must have a + b + c = 12. This eliminates the first possibilities arising from the first and second equations. The latter possibilities satisfy this condition and produce the same triplet (a,b,c). Namely, (3,4,5).

F96S23 Assuming $2^{10} = 10^3$, we have $2 = 10^{.3}$ or log 2 = .3. Since $4 = 2^2$, log 4 = 2log2 = .6 and since 5 = 10/2, log 5 = log 10 - log 2 = 1 - .3 = .7 . To compute log 3, we use the property of log x, as seen from its nonlinear concave downward graph, that for any a,b,c > 0
with a ≤ b ≤ c, log b > 1/2 (log a + log c). That is, log 3 > 1/2 (log 2 + log 4) = 1/2 (.3 + .6) = 4.5. Therefore, log 3 = .5. With this, log 6 = log 2 + log 3 = .3 + .5 = .8 .
Thus, the logarithms in order are .3, .5, .6, .7 and .8.

F96S24 $x^3 - 36x + y = (x - 6)x(x + 6) + y = 0$. Hence, (x - 6)x(x + 6) = -y.
Searching for the largest y, we can assume y is positive. Therefore, the product on the left must be most negative, greater than -1000, and integral. This will occur for x near -10. Testing for x = -10, -11 and -12, we find that x = -11 producing the product -935 satisfies the criteria. Therefore, y = 935.

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F96S25 For $x = 0$, y is easily seen to be 1. For all positive integers, $(x+3)!$ is a multiple of 4 and, therefore, the argument of the cosine is a multiple of 2π and the cosine is 1. Thus, $y = 0$ for all positive integers. The range is the set $\{0,1\}$.

F96S26 The power of 10 that appears in the scientific notation representation of an integer greater than 1 is always one less than the number of digits in the integer. C(100) = 12345678910111213...9899100 has 192 digits. Therefore, the power of 10 is 191.

F96S27 This binomial is the difference of two squares and, since $65{,}536 = 2^{16}$, factoring will produce a factor which is also a difference of two squares. This will occur repeatedly until we find factors of $3 - 2 = 1$ and $3 + 2 = 5$, which must be its smallest prime factor.

F96S28 $x + \frac{1}{x} = 2\cos 15°$ is equivalent to the quadratic $x^2 - 2x\cos 15° + 1 = 0$ whose roots are the complex conjugates $\cos 15° \pm i \sin 15°$. By DeMoivre's Theorem, $x^4 = \cos 60° \pm i\sin 60°$ and $x^{-4} = \cos 60° \mp i\sin 60°$.
The sum is therefore, $2\cos 60° = 2(\frac{1}{2}) = 1$.
An alternative solution:

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 4\cos^2 15°$$

$$x^2 + \frac{1}{x^2} = 2(2\cos^2 15° - 1) = 2\cos 30° = \sqrt{3}$$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2 = 3$$

$$x^4 + \frac{1}{x^4} = 1$$

F96S29 The expression $\dfrac{\frac{p}{q} - \frac{q}{p}}{p - q}$ is equivalent to $\dfrac{p + q}{pq}$. The sum of the roots is $\sqrt{51}/\sqrt{17}$, the product of the roots is $\sqrt{3}/\sqrt{17}$, and the quotient we seek is $\sqrt{17}$.

F96S30 Consider any one of the squares and suppose it has side x. The radius of the inscribed circle is $x/2$ and the area of the shaded region within the square, but outside its inscribed circle, is $x^2(1 - \pi/4)$ and the side of the square inscribed in the inscribed circle is $x/\sqrt{2}$. It is readilly seen that the sides of the successive squares with an outermost square of side s are $s, \frac{s}{\sqrt{2}}, \frac{s}{2}, \frac{s}{2\sqrt{2}}, \frac{s}{4}, \ldots$ and that the n^{th} square has side $\dfrac{s}{\sqrt{2}^{\,n-1}}$. Therefore, the sum of the shaded regions is the infinite series

$$\left(1 - \frac{\pi}{4}\right)s^2 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

$$\text{or } \left(1 - \frac{\pi}{4}\right)s^2\,(2)$$

$$\text{or } \left(2 - \frac{\pi}{2}\right)s^2$$

For $s = \sqrt{2}$, the sum is $4 - \pi$.