New York City Interscholastic Mathematics League

JUNIOR DIVISION | CONTEST NUMBER ONE | FALL 1996

PART I: 10 Minutes NYCIML Contest One Fall 1996

F96J1. A pyramid has a regular polygon as its base. The pyramid has a total of 160 edges. Compute the number of sides in the base of the pyramid.

F96J2. Compute the value of x if $(x!)^2 = \dfrac{(1996!)^2 - (1995!)^2}{(1996)^2 - 1}$.

PART II: 10 Minutes NYCIML Contest One Fall 1996

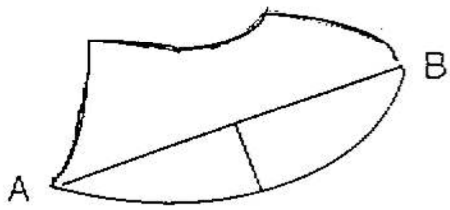
F96J3. Compute the units digit of $(1997)^{1996}$.

F96J4. The roots of $x^2 + bx + c = 0$ (where $b,c \neq 0$) are b and c. Compute the ordered pair (b,c).

PART III: 10 Minutes NYCIML Contest One Fall 1996

F96J5. When reduced to simplest form, $\dfrac{(2^3-1)(3^3-1)(4^3-1)\cdots(10^3-1)}{(2^3+1)(3^3+1)(4^3+1)\cdots(10^3+1)}$ can be written as $\frac{a}{b}$. Compute the value of a+b.

F96J6. Last summer, a circular table was damaged in Hurricane Bertha. In order to calculate the length of the diameter of the original table, a 12 inch ruler was placed forming chord $\overline{AB}$. The midpoint of the ruler was found to be one inch from the circumference of the table. Compute the number of inches that were contained in the diameter of the table.



Answers

1. 80	3. 1	5. 92
2. 1995	4. (1,−2)	6. 37

New York City Interscholastic Mathematics League

JUNIOR DIVISION CONTEST NUMBER TWO FALL 1996

PART I: 10 Minutes NYCIML Contest Two Fall 1996

F96J7. When written in simplest form, the fraction $\frac{1994!+1995!}{1996!}$ is $\frac{a}{b}$. Compute the value of a+b.

F96J8. In $\triangle ABC$, $m\angle A=45$, $m\angle ABC=105$, and the length of altitude $\overline{BD}$ is 20. Compute the length of the longest side of $\triangle ABC$ to the *nearest tenth.*

PART II: 10 Minutes NYCIML Contest Two Fall 1996

F96J9. In $\triangle ABC$, AB= 13 and BC = 15. Altitude $\overline{BD}$ and median $\overline{AE}$ are drawn and BD =12, find the area of $\triangle ABE$.

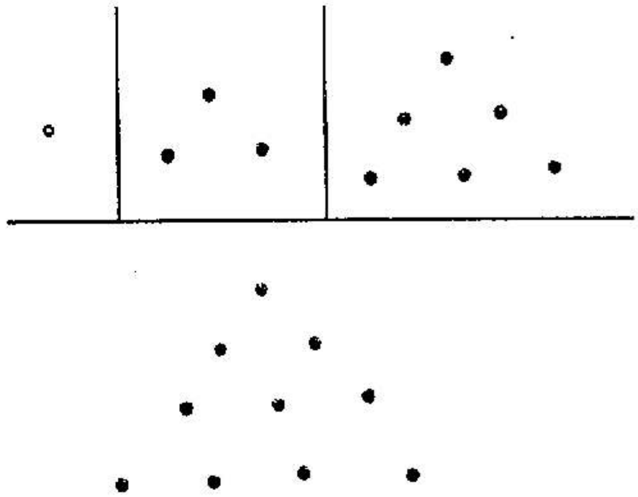
F96J10. Compute the ordered pair (x,y) that satisfies the following pair of equations: (where n! represents n factorial)

$$2x! - 5y! = 120$$
$$3x! - 10y! = 120.$$

PART III: 10 Minutes NYCIML Contest Two Fall 1996

F96J11. Three circles are mutually tangent externally. All three are tangent to the same line and lie on the same side of the line. The two bigger circles are congruent and the smaller circle has radius 1. Compute the radius of the two larger circles.

F96J12. The first four "triangular" numbers are 1,3,6 and 10. (They are shown geometrically on the right.) If the k^{th} triangular number is 4950, compute the value of k.



Answers		
7. 1996	9. 42	11. 4
8. 54.6	10. (5,4)	12. 99

New York City Interscholastic Mathematics League

JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 1996**

PART I: 10 Minutes NYCIML Contest Three Fall 1996

F96J13. If $x = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}$, compute the value of x to the *nearest hundredth*..

F96J14. Compute the value of the sum $\frac{100!}{99!} + \frac{99!}{98!} + \frac{98!}{97!} + \cdots + \frac{2!}{1!} + \frac{1!}{0!}$.

PART II: 10 Minutes NYCIML Contest Three Fall 1996

F96J15. In ΔABC, medians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect at G. If the area of ΔABC is 24, find the area of quadrilateral BFGD.

F96J16. Compute the value of $x^2 + y^2$ if (x,y) is a solution of the system of equations: $xy=5$ and $x^2y + 2x = xy^2 + 2y + 77$.

PART III: 10 Minutes NYCIML Contest Three Fall 1996

F96J17. If $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \cdots (1 - \frac{1}{100^2}) = \frac{a}{b}$, where a and b are relatively prime integers, compute the value of b – a.

F96J18. Two different squares can be inscribed in any isosceles right triangle. Compute the ratio of the area of the larger square to the area of the smaller square in lowest terms.

Answers

13. 0.73	15. 8	17. 99
14. 5050	16. 131	18. 9:8

New York City Interscholastic Mathematics League

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **FALL 1996**

Solutions

F96J1. After drawing diagrams for a few such pyramids, one should see that the number of edges in the pyramid is twice the number of sides in the base. Thus the base has 80 sides.

Answer: 80

F96J2. $(x!)^2 = \frac{(1996!)^2-(1995!)^2}{(1996)^2-1} = \frac{(1995!)^2((1996)^2-1)}{(1996)^2-1} = (1995!)^2$. So $x = 1995$

Answer: 1995

F96J3. A number, x, ending in "7" has the following units digit:

Power of x	Units Digit
x^0	1
x^1	7
x^2	9
x^3	3
x^4	1
x^5	7

Power of x	Units Digit
x^6	9
x^7	3
x^8	1
x^9	7
x^{10}	9
...	

The pattern indicates that all one needs to do is reduce the exponent modulo 4 and use the beginning of the table. Thus $(1997)^{1996}$ will have the same units digit as x^0 or, 1

Answer: 1

F96J4. Since b and c are roots of the equation, we have $(x-b)(x-c) = x^2 + bx + c$. This means that $bc = c$ so that $b = 1$. Also, $b+c = -b$ so that $1+c = -1$ giving $c = -2$.

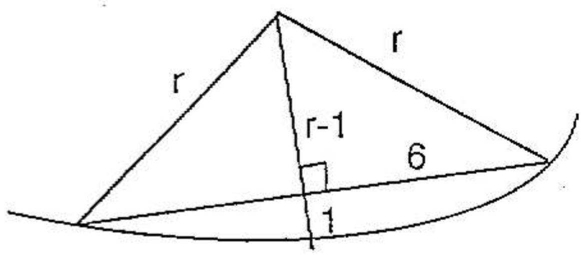
Answer: (1,−2)

F96J5. Note that $x^3 - 1 = (x-1)(x^2+x+1)$ and $x^3 + 1 = (x+1)(x^2-x+1)$.

The desired fraction is $\frac{(2-1)(2^2+2+1)\cdot(3-1)(3^2+3+1)\cdot(4-1)(4^2+4+1)\cdots(10-1)(10^2+10+1)}{(2+1)(2^2-2+1)\cdot(3+1)(3^2-3+1)\cdot(4+1)(4^2-4+1)\cdots(10+1)(10^2-10+1)} =$

$\frac{1\cdot2\cdot3\cdot4\cdots9}{3\cdot4\cdot5\cdot6\cdots11}\cdot\frac{7\cdot13\cdot21\cdot31\cdot43\cdot57\cdot73\cdot91\cdot111}{3\cdot7\cdot13\cdot21\cdot31\cdot43\cdot57\cdot73\cdot91} = \frac{1\cdot2}{10\cdot11}\cdot\frac{111}{3} = \frac{37}{55}$. Thus $a+b = 37+55 = 92$.

Answer: 92



F96J6. Let r = the length of the radius. The Pythagorean Theorem gives $6^2 + (r-1)^2 = r^2$. This is equivalent to $36 + r^2 - 2r + 1 = r^2$ so that $2r = 37$. Thus the diameter has length 37.

Answer: 37

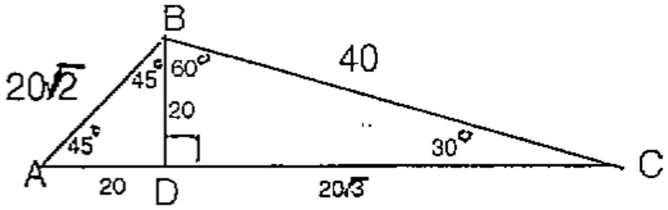
Please note: Concepts used today will be repeated later this year.

JUNIOR DIVISION | **CONTEST NUMBER TWO** | **FALL 1996**

Solutions

F96J7. $\frac{1994!+1995!}{1996!} = \frac{1994!(1+1995)}{1996\cdot1995\cdot1994!} = \frac{1}{1995}$. Thus a=1 and b = 1995 and a+b = 1996.

Answer: 1996

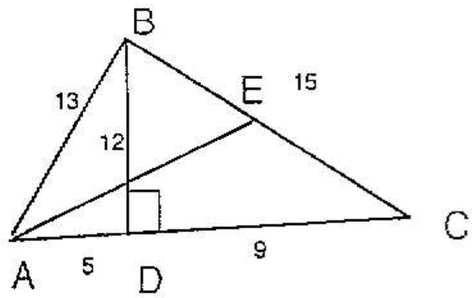


F96J8. AC= $20 + 20\sqrt{3} = 20(1+\sqrt{3}) \approx 20(2.732) = 54.64$.

Answer: 54.6

F96J9. Using the Pythagorean Theorem twice, AD= 5 and DC = 9. Thus, the area of ΔABC is $\frac{1}{2}(14)(12) = 84$. Median $\overline{AE}$ cuts the triangle into two triangles of equal area. This means that the area of $\Delta ABE = \frac{1}{2}(84) = 42$

Answer: 42

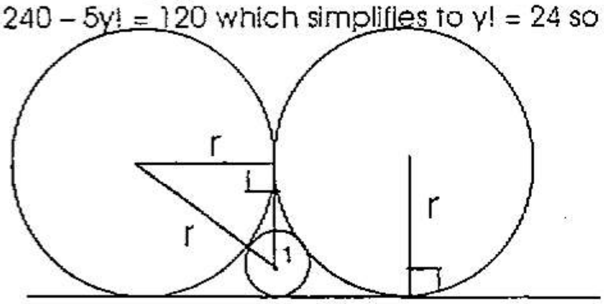


F96J10. The given system is equivalent to:

$$4x! - 10y! = 240$$
$$3x! - 10y! = 120.$$

Subtracting gives x! = 120 so x = 5. Substitute in the first equation that was given: 240 – 5y! = 120 which simplifies to y! = 24 so that y = 4.

Answer: (5,4)



F96J11. The right triangle shown has lengths r, r–1, and r+1 so that $r^2 + (r-1)^2 = (r+1)^2$ so that

$2r^2 - 2r + 1 = r^2 + 2r + 1$ or $r^2 - 4r = 0$ meaning r = 4.

Answer: 4

F96J12.

n	Triangular #
1	1
2	3
3	6

n	Triangular #
4	10
5	15
6	21

The n^{th} triangular number can be represented by $\frac{n(n+1)}{2}$. (or ${}_{n+1}C_2$). (This can be derived in several ways). This means $\frac{k(k+1)}{2} = 4950$ giving k = 99.

Answer: 99

New York City Interscholastic Mathematics League

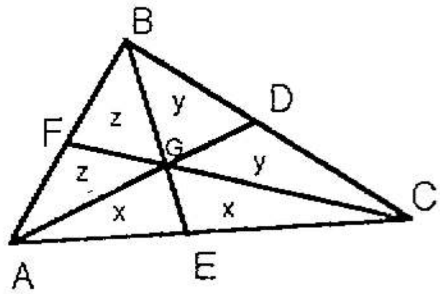
JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 1996**

Solutions

F96J13. Let $x = \frac{2}{2+x}$. Thus $x^2 + 2x - 2 = 0$ and $x = \frac{-2 \pm \sqrt{4+8}}{2}$. Since x is positive, we get $x \approx -1+\sqrt{3} = -1 + 1.732$. To the nearest hundredth this is 0.73. <u>Answer:</u> 0.73

F96J14. $\frac{100!}{99!} + \frac{99!}{98!} + \frac{98!}{97!} + \cdots + \frac{2!}{1!} + \frac{1!}{0!} = (100+99+98+ \cdots +2+1) = \frac{100}{2} \cdot (101) = 5050$
<u>Answer:</u> 5050

F96J15. <u>Theorem:</u> The six disjoint triangles formed when the three medians are drawn are equal in area. (A median cuts a triangle into two triangles of equal area, so in the diagram, we get $x+2z = x+2y$ or $z=y$. Likewise, we get $y + 2z = y + 2x$ so $z=x$. Since $x = y = z$, the needed area is $\frac{1}{3}(24) = 8$.
<u>Answer:</u> 8



F96J16. The second equation can be rewritten as $xy(x-y) + 2(x-y) = 77$. Thus $(xy+2)(x-y) = 77$. Since $xy = 5$, we get $x-y = 11$.
Squaring gives $x^2 - 2xy + y^2 = 121$ or $x^2 + y^2 = 131$. <u>Answer:</u> 131

F96J17. The expression equals $(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{4}) \cdots (1-\frac{1}{100})(1+\frac{1}{100})$ $= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots \frac{98}{99} \cdot \frac{100}{99} \cdot \frac{99}{100} \cdot \frac{101}{100} = \frac{1}{2} \cdot \frac{101}{100} = \frac{101}{200}$ so that $a=101$ and $b=200$. Thus $b - a = 99$. <u>Answer:</u> 99

F96J18. Let x = the length of a leg of the isosceles right Δ. In the first case, using the smaller isosceles right Δ created, each side of the square has length $\frac{x}{2}$. Thus the area of this square is $\frac{x^2}{4}$. In the second case, the hypotenuse of the original Δ has length $x\sqrt{2}$. Using the two smaller isosceles right Δs, each side of the square has length $\frac{x}{3}\sqrt{2}$. This square has area $\frac{2x^2}{9}$. The required ratio is $\frac{x^2}{4} \div \frac{2x^2}{9}$. This is equivalent to $\frac{x^2}{4} \cdot \frac{9}{2x^2} = \frac{9}{8}$. <u>Answer:</u> 9:8