NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER ONE SPRING 1996

PART I: 10 Minutes NYCIML Contest One Spring 1996

S96J1. On March 1st, Wenhua started on a bike trip traveling one mile that day. On March 2nd, she traveled two miles, on March 3rd three miles, on March 4th four miles and so on. On March 5th, Pablo started from the same place and traveled exactly 9 miles each day. On what date did Pablo first catch up with Wenhua?

S96J2. On Planet X the gravity results in coin tosses having only three equally likely possibilities: heads, tails, or lands on edge. Someone flipped three coins on Planet X. Compute the probability that exactly two coins land on edge.

PART II: 10 Minutes NYCIML Contest One Spring 1996

S96J3. Compute $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$

S96J4. Compute the value of xy if $x + y = 26$ and

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{13}{6}$$

PART III: 10 Minutes NYCIML Contest One Spring 1996

S96J5. Solve for all real values of x: $x^4 + 4x^2 = 32$.

S96J6. Compute the greatest number of different products that can be created using different prime numbers less than 42, each product having the same number of factors.

Answers

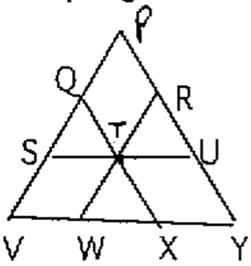
1. March 8th	3. 2	5. −2 and 2
2. $\frac{2}{9}$	4. 144	6. 1716

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER TWO SPRING 1996

PART I: 10 Minutes NYCIML Contest Two Spring 1996

S96J7. ΔPVY is equilateral with points Q,R,S,U,W, and X trisecting the sides of the triangle. T is the common point of intersection of $\overline{QX}$, $\overline{SU}$, and $\overline{RW}$. Of the points, P,Q,R,S,T,U,V, W,X, and Y, how many sets of three are the vertices of equilateral triangles?



S96J8. Jack and Jill have the same number of horses. Jack can make twice as many different teams of horses using three horses each than Jill can make using two horses each. Compute the total number of horses that Jack and Jill have together.

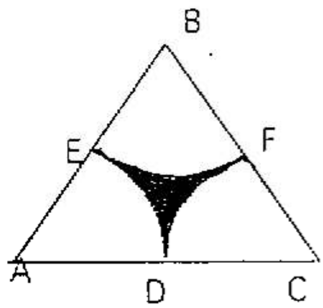
PART II: 10 Minutes NYCIML Contest Two Spring 1996

S96J9. The product of three real numbers in geometric progression is 64. The sum of their cubes is 584. Compute the sum of the numbers.

S96J10. If $x < y < z$, where x, y and z are positive integers, the sum of whose reciprocals is an integer, compute the sum $x + y + z$.

PART III: 10 Minutes NYCIML Contest Two Spring 1996

S96J11. The shaded region DEF is formed by circles of radius one drawn with centers A,B and C of equilateral ΔABC. If AC = 2, compute the area of shaded region DEF.



S96J12. If $\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{8} \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{w} = \frac{11}{24} \\ \frac{1}{x} + \frac{1}{z} + \frac{1}{w} = \frac{1}{2} \\ \frac{1}{y} + \frac{1}{z} + \frac{1}{w} = \frac{13}{24} \end{cases}$ compute $(x+y)(z+w)$.

Answers

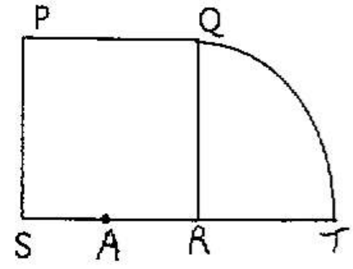
7. 13	9. 14	11. $\sqrt{3} - \frac{\pi}{2}$ or $\frac{2\sqrt{3} - \pi}{2}$
8. 16	10. 11	12. 200

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER THREE SPRING 1996

PART I: 10 Minutes NYCIML Contest Number Three Spring 1996

S96J13. PQRS is a unit square. At A, the midpoint of $\overline{SR}$, a circle is constructed with radius $\overline{AQ}$. The circle intersects $\overleftrightarrow{SR}$ at T. Compute the length of $\overline{ST}$.



S96J14. The sum of two positive numbers, multiplied by the second number is 84. The difference of the two numbers, multiplied by the first number is 16. Compute the largest possible sum of the two numbers.

PART II: 10 Minutes NYCIML Contest Three Spring 1996

S96J15. Four of the vertices of a cube, each of whose sides has length one, are connected to form a *regular* tetrahedron(a solid with four faces, each of which is an equilateral triangle). Compute the surface area of the tetrahedron.

S96J16. Compute the sum of all real x such that $x^{x^2-3x+2} = 1$.

PART III: 10 Minutes NYCIML Contest Three Spring 1996

S96J17. Compute the product xyz if x, y, and z are real and

$$\begin{cases} x + y + z = 6 \\ x^2 + y^2 + z^2 = 14 \\ x^2 + z^2 = 10 \end{cases}$$

S96J18. Compute the product of the non-zero digits in the sum

$$1^3 + 2^3 + 3^3 + \ldots + 75^3$$

Answers

13. $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1+\sqrt{5}}{2}$	15. $2\sqrt{3}$	17. 6
14. 14	16. 2	18. 160

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — SPRING 1996

Solutions

S96J1. *Method I:*

Date	Wenhua's Total Distance	Pablo's Total Distance
March 1	1	–
March 2	3	–
March 3	6	–
March 4	10	–

Date	Wenhua's Total Distance	Pablo's Total Distance
March 5	15	9
March 6	21	18
March 7	28	27
March 8	36	36

Method II: Let x = the number of days Wenhua has been traveling. Wenhua's distance is therefore $\frac{1}{2}\bullet x(x+1)$ and Pablo's distance, starting March is $9(x-4)$. The equation $\frac{1}{2}\bullet x(x+1) = 9(x-4)$ has roots 8 and 9. (Both have traveled the same distance at the end of March 8th and March 9th.) Since the date Pablo first caught up is required, the answer is March 8th. <u>Answer:</u> March 8th

S96J2. The desired probability is ${}_3C_2 \bullet \left(\frac{1}{3}\right)^2 \bullet \left(\frac{2}{3}\right) = \frac{2}{9}$ <u>Answer:</u> $\frac{2}{9}$

S96J3. Let $x = \sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$. This means that $x = \sqrt{2+x}$ or $x^2 = 2+x$ so that $(x-2)(x+1) = 0$. Since x is positive, we reject $x = -1$ meaning $x = 2$. <u>Answer:</u> 2

S96J4. Let $a = \frac{x}{y}$. Now we can eliminate a variable! We get $\sqrt{a} + \sqrt{\frac{1}{a}} = \frac{13}{6}$. Squaring both sides and simplifying leads to $36a^2 - 97a + 36 = 0$. Solve by factoring to get $a = \frac{4}{9}$ or $a = \frac{9}{4}$. If $a = \frac{4}{9}$, $x = \frac{4}{9}y$. Substituting in $x+y = 26$ gives $x = 8$ and $y = 18$. Thus $xy = 144$. If $a = \frac{9}{4}$ similar analysis gives $x = 18$ and $y = 8$ and $xy = 144$.

<u>Answer:</u> 144

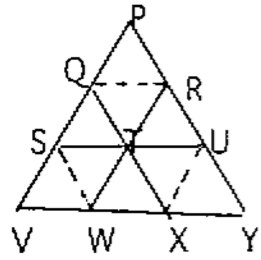
S96J5. The equation is "biquadratic" and can be solved by factoring: $x^4 + 4x^2 - 32 = 0$. This gives: $(x^2+8)(x^2-4) = 0$ which has only two real roots: $x = -2$ and $x = +2$. <u>Answer:</u> -2 and $+2$

S96J6. There are 13 primes less than 42: 2,3,5,7,11,13,17,19,23,29,31,37 and 41. If the number is to have 5 factors, there are ${}_{13}C_5 = 1287$ possible products. If the number is to have 6 factors, there are ${}_{13}C_6 = 1716$ possible products. If the number is to have 7 factors, there are ${}_{13}C_7 = 1716$ possible products. If the number is to have 8 factors, there are ${}_{13}C_8 = 1287$ possible products. The greatest number of products is 1716 using 6 or 7 factors. <u>Answer:</u> 1716

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER TWO SPRING 1996

Solutions



S96J7. There are 9 disjoint "small" equilateral triangles in the diagram. There are 3 "big" equilateral triangles, each consisting of 4 smaller ones. In addition, the original triangle is equilateral. This gives a total of 13 equilateral triangles.

Answer: 13

S96J8. Let x = the number of horses that Jack has. The problem implies that ${}_xC_3 = 2({}_xC_2)$. This leads to $\frac{x(x-1)(x-2)}{3!} = 2\frac{x(x-1)}{2!}$ which gives x= 8. Jack and Jill EACH have 8 horses.

Answer: 16

S96J9. Let the numbers be represented by x, ax, and a^2x. The product is 64 meaning $a^3x^3 = 64$. so ax = 4. Thus the numbers are x, 4 and a^2x. The sum of cubes is 584 meaning $x^3 + 64 + a^6x^3 = 584$. This means that $x^3(1+a^6) = 520$ so that $\frac{64}{a^3}(1+a^6) = 520$ and $\frac{8}{a^3}(1+a^6) = 65$. Thus $8 + 8a^6 = 65a^3$. $8a^6 - 65a^3 + 8 = 0$. Let $y = a^3$. We have $8y^2 - 65y + 8 = 0$ so $(8y-1)(y-8) = 0$. meaning $y = \frac{1}{8}$ or $y = 8$. Thus $a = \frac{1}{2}$ or $a = 2$. The three numbers are 8,4 and 2 or 2,4, and 8. The sum of the three numbers must be 14.

Answer: 14

S96J10. Let $n = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} < 1 + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$. Note that sin x < y < z, $\frac{1}{x}$ is the largest of the three reciprocals. $\frac{1}{x} < n = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} < \frac{3}{x}$ and $n < \frac{3}{x}$ so that $nx < 3$ or $x < \frac{3}{n}$ so that $1 < x < 3$ This means that x = 2 and n = 1. At this point, we have $\frac{1}{y}+\frac{1}{z} = \frac{1}{2}$ so that $\frac{1}{y} < \frac{1}{2}$ and y < z so that $\frac{1}{z} < \frac{1}{y}$ and $\frac{1}{y}+\frac{1}{y} > \frac{1}{2}$ or $\frac{2}{y} > \frac{1}{2}$. Thus $\frac{1}{y} < \frac{1}{2} < \frac{2}{y}$ or $y > 2 > \frac{y}{2}$ giving y = 3. Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, we get z = 6. This gives x + y + z = 2 + 3 + 6 = 11.

Answer: 11

S96J11. To get the area of an equilateral triangle, we must use the formula $A = \frac{s^2}{4}\sqrt{3}$ which means the area of $\Delta ABC = \sqrt{3}$. Each sector is $\frac{1}{6}$ of a circle. Thus the total circular area is $\frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$. To get the needed area, subtract to get $\sqrt{3} - \frac{\pi}{2}$.

Answer: $\sqrt{3} - \frac{\pi}{2}$ or $\frac{2\sqrt{3} - \pi}{2}$

S96J12. Replace $\{\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w}\}$ by {a,b,c,d} and use 24ths to get:

a+b+c+	= $\frac{9}{24}$
a+b+ +d	= $\frac{11}{24}$
a+ +c+d	= $\frac{12}{24}$
b+c+d	= $\frac{13}{24}$

This means x = 12, y = 8, z = 6 and w = 4 so (x+y)(z+w) =

Answer: 200

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER THREE SPRING 1996

Solutions

S96J13. By the Pythagorean Theorem, AQ $\frac{1}{2}\sqrt{5}$ ST = $\frac{1}{2}\sqrt{5} + \frac{1}{2} = \frac{1}{2}(1+\sqrt{5})$ which happens to be the Golden Ratio! **Answer:** $\frac{1+\sqrt{5}}{2}$

S96J14. Let x = the first number; Let y = the second number

$y(x+y)=84$ $\quad xy + y^2 = 84$ $\quad$ Adding gives $\boxed{x^2 + y^2 = 100}$

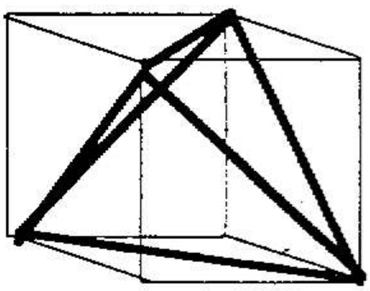
which means

$x(x-y)=16$ $\quad x^2 - xy = 16$

We can divide both sides of the orignal second equation by x:

$x - y = \frac{16}{x}$ so $y = x - \frac{16}{x}$ so $y = \frac{x^2-16}{x}$ so $y^2 = \frac{x^4-32x^2+256}{x^2}$. Add x^2 to both sides of the equation to get $x^2 + y^2 = \frac{2x^4-32x^2+256}{x^2} = 100$. This results in the biquadratic equation $2x^4 - 132x^2 + 256 = 0$ so $x^4 - 66x^2 + 128 = 0$ giving $(x^2-2)(x^2-64) = 0$ so that $x=\sqrt{2}$ and $y=-7\sqrt{2}$ (Reject, since y is not positive) *OR* x= 8 and y = 6. The largest sum x+y istherefore 14. . Answer: 14

S96J15. Each face of the tetrahedron is an equilateral triangle, the length of whose side is $\sqrt{2}$. The area of each face is then computed using $A=\frac{1}{4}s^2\sqrt{3} = \frac{1}{4}(\sqrt{2})^2\sqrt{3} = \frac{1}{2}\sqrt{3}$. Multiply by 4 giving the total area, $2\sqrt{3}$. Answer: $2\sqrt{3}$



S96J16. One way a power can be 1 is for the exponent to be zero. This yields the equation $x^2-3x+2=0$ so x = 1, or x = 2. Another way for a power to be 1 is for the base to be 1 or −1. Since x = 1 arose in the case above, we need not reconsider it. If x=−1, $(-1)^{1+3+2} = 1$ which means x=−1 satisfies the equation. Thus the sum of the roots is −1+1+2 = 2. Answer: 2

S96J17. Subtracting the third equation from the second gives $y^2=4$ so y=−2 or y=+2. The first value does not give real values for x and z. Substitute y=2 to get x+z = 4 and $x^2+z^2 = 10$. Substitute x = 4−z in the second equation to get two ordered triples solving the original system:(1,2,3) or (3,2,1). In both cases, the product xyz is 6. Answer: 6

S96J18. Expression	Sum
1^3	1^2
$1^3 + 2^3$	3^2
$1^3 + 2^3 + 3^3$	$6^2 = (\frac{3\cdot4}{2})^2$
$1^3 + 2^3 + 3^3 + 4^3$	$10^2 = (\frac{4\cdot5}{2})^2$
$1^3 + 2^3 + 3^3 + 4^3 + 5^3$	$15^2 = (\frac{5\cdot6}{2})^2$
• • •	• • •
$1^3 + 2^3 + 3^3 + \ldots + n^3$	$(\frac{n(n+1)}{2})^2$

Using the above generalization, we get $1^3 + 2^3 + 3^3 + \ldots + n^3 = (\frac{75\cdot76}{2})^2 = 2850^2 = 8,122,500$. The product of the non-zero digits is 160. Answer: 160

May 10, 1996

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1996 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Senior A	S96S07	add "with no logarithmic functions"
	S96S30	any ordered triple of the form ((b+3)/2, b, -b)
Senior B	S96B14	add "with no logarithmic functions"
Junior	S96J07	15

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML