NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER ONE

FALL, 1995 Compute the number of ordered pairs of integers (x,y) which satisfy the simultaneous F95S1 inequalities y - x < 0 $y+\frac{1}{2}x>0$

Two runners traveling in opposite directions at the same rate are at the same point on an east-F95S2 west bridge 300 meters long exactly at noon. As one of the runners reaches one end of the bridge a roller blader begins her journey across the bridge and reaches the other side at the same time as the other runner. If the roller blader was traveling at a rate 11/2 times as fast as the runners, how far from the closer end of the bridge were the runners at noon?

PARTII TIME: 10 MINUTES

TIME: 10 MINUTES

FALL, 1995

If A and B are complementary acute angles, compute F95S3 tan A + tan B csc(360 - 2A)

If $a^4b^3 - a^5b^2 = 4050$, compute the ordered pair of positive integers (a,b). F95S4

PARTIII TIME: 10 MINUTES

FALL, 1995

Compute the real number x that satisfies $log_48*log_812*log_{12}16 = log_{16}x$ F95S5

Compute the positive x-intercept of the line that is 3 units from the point (3,0) and passes F9556 through the point (3.6).

ANSWERS

PARTI

1.3

2. 50 meters

3. -2

4. (3,5)

5. 256

 $6.3 + 2\sqrt{3}$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER TWO

| PARTI | TIME: 10 MINUTES | FALL, 1995 | | | |
|----------|--|------------------|--|--|--|
| F95S7 | Compute the real number that satisfies $2\sqrt{x} - 8 + 3x = 0$. | | | | |
| F95S8 | The squares of the first three terms of an increasing arithmetic progression of different real numbers form a geometric progression. Compute the first term of the arithmetic progression if the sum of the cubes of the first three terms is 120. | | | | |
| PART II | TIME: 10 MINUTES | FALL, 1995 | | | |
| F95S9 | Compute the real solutions to $ 2x - 3 + 2x^2 = 5$. | | | | |
| F95S10 | If $ \sin x + \cos x \neq 0$, $\sin 2x = a$ and $b = (\sin x + \cos x)^3$, we expressing b in terms of a. | rite an equation | | | |
| PART III | TIME: 10 MINUTES | FALL, 1995 | | | |
| F95S11 | Compute the number of even divisors of 425 | * | | | |
| F95S12 | Circle O is inscribed in equilateral triangle ABC which is inscribed in square ADEF. If the ratio of the area of circle O to the area of square ADEF in simplest form is π(a + b), compute the ordered pair (a,b). | | | | |

ANSWERS

7. 16/9 8. 2 - 2√2

9.
$$\frac{-1+\sqrt{17}}{2}, \frac{1-\sqrt{5}}{2}$$

10. $b = (1+a)^{3/2}$

11. 180.

12. (2/3, √3/3)

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER THREE

| PARTI | TIME: 10 MINUTES | FALL, 1995 | | |
|----------|---|---|--|--|
| F95S13 | Let d(x) represent the number of zeros appearing at the end of positive integer x. Compute d(19!) - d(9!) | | | |
| F95S14 | A circular wheel is positioned in the interior of a circular track so coincides with point B of the track. The wheel begins to travel alon the area of the wheel to the area of the region within the track is 9/2 complete laps around the track made by the wheel the first time the again. | rel begins to travel along the track. If the ratio of a within the track is 9/25, compute the number of | | |
| PART II | TIME: 10 MINUTES | FALL, 1995 | | |
| F95S15 | Compute $\sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$ | | | |
| F95S16 | How many positive integers x, $x < 100$, exist for which $3^x - x^3$ is divisible by 6? | | | |
| | | 39 | | |
| PART III | TIME: 10 MINUTES | FALL, 1995 | | |
| F95\$17 | Compute the values of positive integer b, $b < 20$, such that the base b square? | numeral 2100 is a perfect | | |
| F95\$18 | Solve for all real numbers, x, that satisfy $\frac{x - [x]}{x + [x]} = \frac{1}{6}$. | | | |
| | ([x] represents the greatest integer less than or equal to x.) | | | |
| | | | | |

ANSWERS

13. 2

14.5

15, 3

16. 17

17. 4 and 12

18. 7/5 and 14/5

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENTOR A DIVISION CONTEST NUMBER FOUR

| | PART I | TIME: 10 MINUTES | FALL, 1995 | | | |
|---|--------------------------------------|---|------------|--|--|--|
| | F95S19 | A woman born in the latest year prior to 1995 that has 41 as a divisor has a child on January 1, 1995. Compute the age of the child during the first year after 1995 that has 41 as a divisor. | | | | |
| | F95S20 | Find the area of the region bounded by the graphs of $ y - x = 3$ and $ 5y + x - 6 = 9$. | | | | |
| _ | - | | | | | |
| | PART II | TIME: 10 MINUTES | FALL, 1995 | | | |
| | bF(a) and $F(p) = p$ for prime p , | | | | | |
| | F95S22 | Compute the number of triangles with integer sides, exactly two congruent sides and a perimeter of 1995. | | | | |
| | | | | | | |
| | PART III | TIME: 10 MINUTES | FALL, 1995 | | | |
| | F95S23 | Write the units digit of the product $17^{95}18^{95}19^{95}$. | | | | |
| | F95S24 | The windsheild of a car is a rectangle 45 inches long by 24 inches high. The wiper blades are each 18 inches long and rotate in the same direction. When at rest, the blades form a straight segment 36 inches long with one endpoint at the lower right vertex of the windsheild. When | | | | |

the windshield that is cleaned by the blades.

rotated to their maximum, the rotating endpoint of the left blade meets the left side of the windshield. Compute, in terms of π , the number of square inches in the area of the region on

ANSWERS

19.14

20.18

21. 7,980

22, 498

23. 4

24. 162π - 81√3

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER FIVE

PART I TIME: 10 MINUTES

FALL, 1995

F95S25

If $\sqrt{16} - 6\sqrt{7} = a + b\sqrt{7}$, compute all the ordered pairs of real numbers (a, b).

F95S26

A runner jogs in a circular fashion as follows: She runs a complete circle of radius 10 meters and then runs 10 meters along a tangent to the point at which she completes the circle. From there she begins to run a new circle with the same center as the original. When she returns to her starting point, she again runs along a tangent line the length of the current radius and repeats the process. Compute the number of square meters in the area of the region bordered by her fifth complete circle.

PART II TIME: 10 MINUTES

FALL, 1995

F95S27

Compute the number of different polyhedra having 8 faces that are equilateral triangles, 4 faces that are squares and 5 faces that are regular pentagons.

F95S28

Six students in the same class hand their schedule cards to their teacher. After reviewing the cards, she puts each of them in a separate envelope and seals them. When returning the cards the next day, she realized that she forgot to write the students names on the envelopes, but returns them at random. Compute the probability that none of the students receives his or her own schedule?

PARTIE

TIME: 10 MINUTES

FALL, 1995

F95529

If $9^{x} + 9^{y} = 198$ and $3^{x} - 3^{y} = 6$, compute x - y.

F95S30

Let S be a set of twelve points that are equally spaced on a circle. Compute the number of different acute triangles that can be formed using three points from S as vertices.

ANSWERS

25.
$$(\sqrt{7}, \frac{3\sqrt{7}}{7})$$
 and $(3,1)$

26. 1600π

27.0

28. 53/144

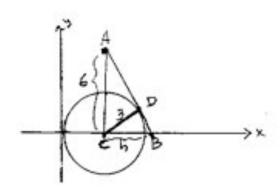
29.4

30, 40

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER ONE SOLUTIONS

FALL 1995

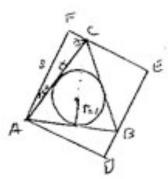
- The graphs of the straight lines y = x, y = -x/2, and x = 3 border the region in the coordinate plane in which we find the solutions. The solutions are the 3 lattice points in the region: (1,0), (2,0) and (2,1).
- Let x be the distance from the closer end of the bridge. The roller blader starts her journey when both
 runners have moved x meters. Therefore, she travels 300 meters in the same time that the runner still
 on the bridge runs 300 2x meters. Since distance and rate vary directly, we have
 300/(300 2x) = 3/2 which yields x = 50 meters.
- 3. The given expression is equivalent to $(\tan A + \cot A)\sin(360-2A) = (\tan A \cot A)\sin 2(180 A)$ $= (\tan A \cot A)(2)\sin(180-A)\cos(180-A)$ $= -2(\tan A + \cot A)\sin A\cos A$ $= -2(\sin^2 A + \cos^2 A)\sin A\cos A$ = -2
- \div . The unique prime factorization of 4050 is $2 \cdot 3^4 \cdot 5^2$. Since a and b are positive integers and $a^4b^3 a^5b^2 = a^4b^2(b-a)$, it must be the case that (a,b) = (3,5).
- 5. Let $4^a = 8$, $8^b = 12$ and $12^c = 16$. Therefore $x = 16^{abc} = 4^{2abc} = (((4^a)^b)^c)^2 = 16^2 = 256$.
- 6. A line that is 3 units from (3,0) is a tangent to the circle (x 3)² + y² = 9. Let h represent the distance from (3,0) to the x-intercept of the line. Since the line must pass through (3,6) we can use the right triangle with vertices C(3,0), A(3,6) and B(3+h,0) with the radius of the circle drawn to the line as the altitude to the hypotenuse. Call their point of intersection D. From right triangle ACD we have AD = √27 and since AC² = (AD)(AB) we get AB = 36/√27 = 4√3. Therefore, h = √12 from the Pythagorean Theorem on right triangle ABC and the x-intercept is 3 + 2√3



NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER TWO SOLUTIONS

FALL 1995

- 7. Let $y = \sqrt{x}$. The equation is, therefore, a simple quadratic: $3y^2 + 2y - 8 = (3y - 4)(y + 2) = 0$. Since y must be positive, the only solution is $y = \sqrt{x} = 4/3$ or x = 16/9.
- 8. Let the first three terms be a d, a and a + d. Since their squares form a geometric progression, we have $a^4 = (a d)^2(a d)^2 = (a^2 d^2)^2$ which yields $d^2(d^2 2a^2) = 0$. Since the terms are different, $d \neq 0$, and we have $d^2 = 2a^2$. The sum of their cubes expands and simplifies to $3a^3 6ad^2$ and by substitution we have $3a^3 + 12a^3 = 15a^3 = 120$ and a = 2. Therfore, $d = \sqrt{8}$ and the first term of the arithmeno progression is $2 2\sqrt{2}$
- The given equation yields the following two auxilliary equations: $2x - 3 = 5 - 2x^2$ and $2x - 3 = 2x^2 - 5$. The four possible solutions are $\frac{1}{2}(-1 \pm \sqrt{17})$ and $\frac{1}{2}(-1 \pm \sqrt{5})$. The first pair yield $2x^2 = 9 = \sqrt{17}$ and the second pair yield $2x^2 = 3 = \sqrt{5}$. Since $5 - 2x^2$ must be nonnegative, we can accept only $2x^2 = 9 - \sqrt{17}$ and $2x^2 = 3 - \sqrt{5}$ from the solutions $x = \frac{1}{2}(-1 + \sqrt{17})$ and $x = \frac{1}{2}(-1 + \sqrt{17})$ and $x = \frac{1}{2}(-1 + \sqrt{17})$. Arswer: $\frac{-1 + \sqrt{17}}{2}$
- 10. $(\sin x + \cos x)^3 = ((\sin x + \cos x)^2)^{3/2} = (\sin^2 x + \cos^2 x + 2\sin x \cos x)^{3/2} = (1 + \sin 2x)^{3/2} = (1 x)^{3/2}$ Therefore, $b = (1+x)^{3/2}$
- 11 $42^5 = 2^5 3^5 7^5$ has 6^3 divisors. The odd divisors are the 36 divisors of $3^5 7^5$. Therefore, 42^5 has 216 36 = 180 even divisors.
- Triangle ABC can be positioned in only one way making the problem well defined. Right triangles ADB and AFC are congruent with acute angles 15 and 75 degrees. The center of a circle inscribed in a triangle lies at the intersection of the angle bisectors which, for an equilateral triangle, are the medians which intersect each other forming segments in the ratio 1:2. The shorter segment is the radius of the inscribed circle. Since we are seeking a ratio, we can assume the radius to be 1. Therefore, the altitude of the equilateral triangle is 3 and the side of the equilateral triangle is $2\sqrt{3}$. The side of the square is, thus, $2\sqrt{3}\cos 15$ and $s^2 = 12\cos^2(15) = 12(1/2)(1+\cos 30) = 6(1+\sqrt{3}/2)$. The ratio we seek is $\pi/(6-3\sqrt{3}) = \pi(6-3\sqrt{3})/9 = \pi(2/3-\sqrt{3}/3)$.



NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER THREE SOLUTIONS

FALL 1995

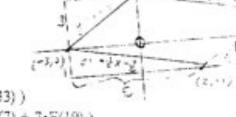
- 13. n! will have as many trailing zeros as there are factors of 10.
 19! = 19*18*17*16*15*14*13*12*11*10*9*8*7*6*5*4*3*2*1
 = 19*18*17*16* 3*14*13*12*11 * 9*8*7*6 * 2*3 *1*(5*2)*10*(5*2)
 Therefore, d(19!) = d(9!) = 3 ! = 2.
- 14. The ratio of the areas yield the ratio of the radii to be 3/5 which is also the ratio of the circumferences. Therefore, each revolution of the wheel covers 3/5 of the track. For the points to coincide after a number of complete laps around the track, this number multiplied by 3/5 must be a positive integer. The least such multiplier of 3/5 is 5.
- 15. Let $x = \sqrt{6} \sqrt{6} \sqrt{6} \dots$ Therefore, $x = \sqrt{6} + x$ or $x^2 = 6 + x$ and x must be positive since it is a square root. The positive solution to the quadratic is x = 3.
- The remainders of 3^{x} when divided by 6 for integers x = 1,2,3,... are all 3. The remainders of x^{2} when divided by 6 for the integers x = 1,2,3,... are respectively 1,2,3,4,5,0,1,2,3,4,5,0,... in this cyclic fashion. Therefore, the remainders of $3^{x} x^{3}$ form the cycle 2,1,0,5,4,3,2,1,0,5,4,3,... showing that $3^{x} x^{3}$ is divisible by 6 for x = 3,9,15,21,27,... = 3 + 6k for k = 0,1,2,3,... These numbers are less than 100 for $0 \le k \le 16$ and, therefore, there are 17 of them.
- A base b number is a perfect square in base b if and only if its base 10 equivalent is a perfect square in base 10. 2100 in base b is equivalent to the base 10 number $2b^3 b^2 = b^2(2b 1)$. This is a perfect square if and only if 2b + 1 is a perfect square. This is the case for b = 4, 12, 24, 40, 60. ... The bases less than 20 are 4 and 12.
- 18. Cross multiplying and simplifying yields the equation x = 7[x]/5. Considering this equation in the specific intervals k ≤ x ≤ k + 1 for integers k we have k ≤ 7k/5 < k + 1 or 0 ≤ 2k/5 < 1. Therefore, there will be solutions for k = 1 and k = 2 only, namely x = 7/5 and 14/5. (k = 0 is an extraneous solution.)</p>

Alternatively, one can graph y = [x]. By determining the slopes of the lines passing through the origin and the endpoints of each of the "steps" and comparing to 5/7, the slope of y = 5x/7, the intersected steps can be ascertained. That is, for x < 0, all the "steps" lie to the right of y = x and, since 5/7 < 1, the line will not intersect any of the steps below the x-axis. For $0 \le x < 1$, the point of intersection occurs at x = 0 which is not a solution. For $x \ge 1$, lines that intersect the first step above the x-axis must have slopes between 1/2 and 1 which is the case for 5/7. The second "step" is intersected by lines with slopes between 1 and 2/3 which is also the case for 5/7. The third "step" is intersected by lines with slopes between 1 and 3/4 which is not the case for 5/7. Successive steps will require slopes nearer to 1. Therefore, the problem reduces to the two auxiliary equations 5x/7 = 1 and 5x/7 = 2 yielding the solutions x = 7/5 and x = 14/5.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE SENIOR A DIVISION CONTEST NUMBER FOUR SOLUTIONS

FALL 1995

- Let the year of the woman's birth be 41x, x is a positive integer. The child's age is therefore 41(x+1) - 1995. The minimum value for x for which this is positive is 49 and the child's age is 14.
- Each equation has a graph consisting of parallel lines. That is, |y x| = 3 produces the lines y = x 3 and y = x 3. Sy + x 6 = 9 produces the lines y = -x/5 3/5 and y = -x/5 + 3. The region bounded by these lines is a parallelogram with vertices (-3,0), (0,3), (5,2), and (2,-1). The area is most easily computed by creating a rectangle with sides parallel to the axes for which the parallelogram is inscribed and subtracting from its area the areas of the four right triangles in the rectangle, but outside the parallelogram. The rectangle has vertices (-3,-1), (-3,3), (5,3) and (5,-1). Therefore, the area of the parallelogram is 32 4/(5 + 9 + 5 + 9) = 32 14 = 18.



- 22. Let the sides of the triangle be the integers a, a and b. We have 2a + b = 1995 and 0 < b < 2a. Therefore, 2a < 1995 < 4a which yields 499 ≤ a ≤ 997. Hence, there are 997 498 = 499 such isosceles triangles. Since we are not including the equilateral triangle with sides 665, the answer is 498.</p>
- 17 for k = 1,2,... produce units digits in the cyclic sequence 7,9,3,1,7,9,3,1,...
 18 for k = 1,2,... produce units digits in the cyclic sequence 8,4,2,6,8,4,2,6,...
 19 for k = 1,2,... produce units digits in the cyclic sequence 9,1,9,1,9,1,9,1,...
 Therefore, the product 17 for k = 1,2,... produce units digits in the cyclic sequence 4,5,4,6,4,6,... establishing that, for k odd, the units digit of the product is 4.
- When the blades are rotated to their maximum, the left blade forms a right triangle with the left side of the windshield with lower side 9 and hypotenuse 18. Their included angle is, therefore, 60° . From this we see that each blade sweeps a third of a circular region each having an area of $324\pi/3 = 108\pi$. There is an overlapping region that is the area of a 60° sector less the area of an equilateral triangle with side 18. This area is $324(\pi/6 \sqrt{3}/4) = 54\pi 81\sqrt{3}$. Therefore, the area of the cleaned region is $216\pi (54\pi 81\sqrt{3}) = 162\pi + 81\sqrt{3}$.

NEW YORK CITY INTERSCHOLASTIC MATH. LFAGLE SENIOR A DIVISION CONTEST NUMBER FIVE SOLITIONS

SOL 17V-1

- 25. Squaring both sides of the governoquation yields 16 + 6√7 = π² + 7b² + 2πb√7. Thus, π² + 7b² + 16 and 2πb = 6. Using n = 3/6 and substituting into the first relationship, we have 7h² + 16h² + 9 = 0. This factors into (7b² = 9)(b² 1) = 0 producing the solutions b = π3√7 and b = π1. The corresponding solutions for a art π² 7 and π3. Note that n + b² 7 must be must we succent as a square root. Thus the acceptable ordered pairs are = (√7, √7) μωσ(3.1).
- 26. The radius of the second circle is 10/2 derived from the susceles right manyle formed by the first radius and the tangent line of equal length. The radius of the third circle can be computed by a smaller process and is found to be 20. The fourth radius will be 20/2 and the fifth will be 40. Therefore, the area of the region bordered by the fifth complete circle is 1600% aquaic maters.



- 27. From the 8 equilateral triangles we would have 26 edges. From the 4 squares we would have 16 edges. Prom the 5 pentagons we would have 25 edges. Thus, we would have a total of 65 edges. However, each edge is shared by exactly two polygons forcing the number of edges to be even. Therefore, there are 0 polyhedra meeting the given specifications.
- 23. Let f(n,k) be the number of ways that each of exactly k out of a students receives his or her own schedule. We seek to compute f(6,0)/6. Clearly, f(n,k) = C_x f(n-k,0) and f(x,0) = xi f(x,x-1) f(x,x-2) f(x,x-3) ... f(x,1)
 Using recursion we can construct the following table:

| - | 5 | * | - | N | - | ž. |
|-----|----|----|---|---|---|----|
| _ | - | H | - | H | H | 0, |
| 0 | - | L | L | L | L | 5 |
| ŭ | 0 | - | | Ĺ | | - |
| 8 | 10 | 0 | - | | | u |
| Į, | 20 | 6 | 0 | - | | 2 |
| 264 | 45 | 60 | w | 0 | | - |
| 365 | 4 | 9 | | _ | 0 | 0 |

Tharefore, the probability we seek is 265/720 or 53/144.



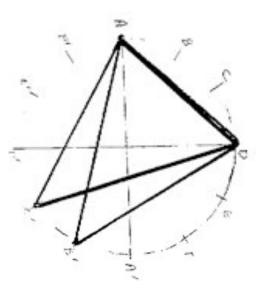
 $9^{4} + 9^{3} \times 3^{3} + 3^{49} \times 19\%$ (3* - 3*) $2^{4} + 3^{44} + 3^{47} + 3^{47} + 6^{2}$. This yields $19\% - 2 + 3^{4/3} - 36$ or $3^{4/9} - 31$. Therefore, n + y = 4.



For an active triangle to be forestal, one of the vertices must be on a different semicide from the other two. By selecting two points for vertices and drawing the diameters from these points, one can identify the possible points for the third vertice as one of the points on the nature are created by the other catopoints of the diameters. The number of such points will equal the number of those on the number are between the selected points. Labeling the points A.B.C.D.B.F.A.Ta.T.C.D.E. and F. the problem is set to be that of identifying the number of different arrangements possible after choosing the first paint.

If A is the liest vertex chosen, then the different triangles will come from AC, AD, AE and AF. The points marked with "will appear as the third vertex chosen. The analysis of triangles produced are respectively 1,2,3 and 4 for a total of 10 including point A. There are 10 different triangles including point B since points A and B cannot form a triangle as they have no points on their musor are. From the 10 points be triangles for point C, we must eliminate any that contain A and B. There is only the 1 we already counted by using A and C. From the 10 including point D, we eliminate the 2 we counted with A and the 1 we counted with B. From the 10 including E, we eliminate the 3 from A, the 2 from B, and the 1 from C. From the 10 for point F, we eliminate all of them as all interior points on the minor area FA', FB', FC' and FD' contain the "points of A, B, C, and D. Therefore, there are 10-6 - (0 + 0 + 1 + 3 + 6 + 10) = 60 - 20 + 40.

The picture below depicts the triangles constructable using points A and D.



Dear Math Team Coach,

Enclosed is your copy of the Fall, 1995 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

| Junior | Question F95J15 was eliminated F95J18 was eliminated | Correct answer You cannot get a remainder of 3 when dividing by 3. It should have read "a + b, b + c, and a + c are all greater than 0". Otherwise imaginary numbers could be considered. |
|----------|--|---|
| Senior A | F95S14 | 3 |
| | F95S24 F95S25 was eliminated | 162π + 81 $\sqrt{3}$ It should have read " a and b are rational". The correct answer would then be (3, 1). Otherwise there are infinite solutions |

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML