NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — FALL 1995

PART I: 10 Minutes — **NYCIML Contest One** — Fall 1995

F95J1. A cube 5 inches on a side, is painted green, and then cut into one inch cubes. Compute the number of small cubes that have not been painted on any face.

F95J2. Compute the area of ΔABC where the coordinates of the vertices of the triangle are $A(0,0)$, $B(1,1)$, and $C(2,3)$.

PART II: 10 Minutes — **NYCIML Contest One** — Fall 1995

F95J3. Compute the value of $\sqrt{\dfrac{\dfrac{1}{3^2}+\dfrac{1}{4^2}}{\dfrac{1}{12^2}-\dfrac{1}{13^2}}}$

F95J4. The number 24! when written out ends with k zeros. Compute the value of k.

PART III: 10 Minutes — **NYCIML Contest One** — **Fall 1995**

F95J5. Solve for all $x > 1$ so that $x\left(\sqrt[x]{x^{15}}\right)=\dfrac{x^x}{x}$

F95J6. When the product $\dfrac{3}{4}\cdot\dfrac{8}{9}\cdot\dfrac{15}{16}\cdot\ldots\dfrac{(n+1)^2-1}{(n+1)^2}\cdot\ldots\cdot\dfrac{399}{400}$ is written in simplest form, the result is $\dfrac{a}{b}$. Compute the value of $b-a$.

Answers

1. 27	**3.** 13	**5.** 5
2. $\frac{1}{2}$	**4.** 4	**6.** 19

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION | **CONTEST NUMBER TWO** | **FALL 1995**

PART I: 10 Minutes | **NYCIML Contest Two** | Fall 1995

F95J7. A cube twelve inches on a side, is painted red, and then cut into one inch cubes. Compute the number of small cubes that have been painted on exactly ONE face.

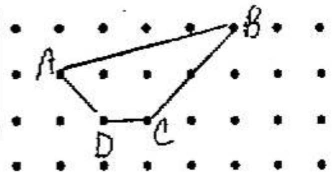
F95J8. An old grandfather forgot how many grandchildren he had; however, he did remember that dividing them into groups of 3, 4, or 5 will always give a remainder of 2. Compute the least number of grandchildren the man can have.

PART II: 10 Minutes | **NYCIML Contest Two** | Fall 1995

F95J9. Compute the number of zeros in which 100! terminates.

F95J10. A student rewrote the square root of $2+\sqrt{3}$ to be the sum $\sqrt{a}+\sqrt{b}$ where a and b are rational numbers in simplest form and a<b. Write the ordered pair (a,b).

PART III: 10 Minutes | **NYCIML Contest Two** | **Fall 1995**



F95J11. Compute the area of quadrilateral ABCD.

F95J12. The pages of a book are numbered consecutively starting from 1. If the digit "1" is used 56 times, compute the number of times the digit "2" is used.

Answers

7. 600	9. 24	11. $3\frac{1}{2}$
8. 62	10. $(\frac{1}{2}, 1\frac{1}{2})$	12. 26

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1995**

PART I: 10 Minutes **NYCIML Contest Three** **Fall 1995**

F95J13. A cube twenty–two inches on a side, is painted yellow, and then cut into one inch cubes. Compute the number of small cubes that have been painted on exactly TWO faces.

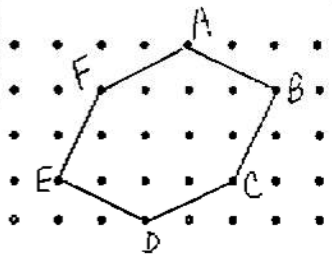
F95J14. Compute the value of the largest integer x such that 2^x divides 24!

PART II: 10 Minutes **NYCIML Contest Three** **Fall 1995**

F95J15. A great–grandmother forgot how many great–grandchildren she had; however, she did remember that dividing them into groups of 3, 10, or 14 will always give a remainder of 3. Compute the least number of great–grandchildren she can have.

F95J16. 22_a, 26_a, and 34_a form a geometric sequence. If "a" is greater than one, compute the value of a, where "a" is the base of the three numbers.

PART III: 10 Minutes **NYCIML Contest Three** **Fall 1995**



F95J17. Compute the area of hexagon ABCDEF.

F95J18. Compute the smallest possible value of a+b+c where a, b and c are distinct integers such that $\sqrt{a+b} + \sqrt{b+c} = \sqrt{a+c}$.

Answers

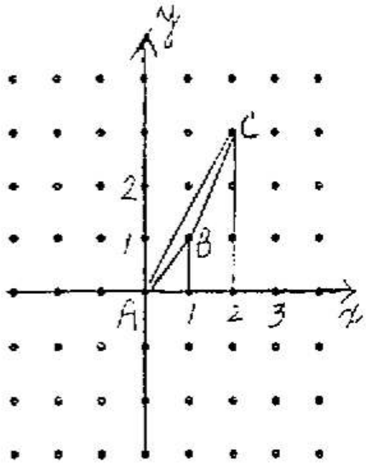
13. 240	**15.** 213	**17.** 12
14. 22	**16.** 7	**18.** 7

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 1995**

Solutions

F95J1. A cube with dimensions 3×3×3 in the interior of the original cube has not been painted. In general, if each edge of a painted cube has been cut into n congruent segments, there are $(n-2)^3$ cubes that are unpainted. **Answer:** 27



F95J2. *Method I:* Connect points to form a right triangle whose area is 3. Subtracting off the areas of the trapezoid (2) and triangle ($\frac{1}{2}$) formed, give the area of $\triangle ABC$ to be $\frac{1}{2}$. *Method II:* The polygon has 3 "lattice points" on its boundary and none in its interior. (A lattice point is a point whose coordinates are integers.) A polygon with no interior lattice points and B lattice points on its boundary (including the vertices) has area $\frac{B}{2} - 1$. This also gives area of $\frac{1}{2}$.

Answer: ½

F95J3. The given expression reduces to $\sqrt{\frac{5^2}{3^2 4^2} \cdot \frac{12^2 13^2}{5^2}}$ which simplifies to 13. **Answer:** 13

F95J4. To get a terminal zero, the factor "2" must pair with a factor "5." Many more terms have 2 as a factor than 5, so concentrate on the 5's: The factors of 24! containing "5" are 5, 10, 15, and 20. Thus there are four zeros in 24!

Answer: 4

F95J5. The equation can be rewritten as $x \cdot x^{15/x} = x^{x-1}$ which simplifies to $x^{15/x+1} = x^{x-1}$. Equating exponents, results in the quadratic equation $x^2 - 2x - 15 = 0$ so $x = 5$ or $x = -3$ (which we reject since the problem called for all <u>positive</u> values of x). **Answer:** 5

F95J6. $\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \ldots \cdot \frac{399}{400} = (1 - \frac{1}{2^2}) \cdot (1 - \frac{1}{3^2}) \cdot (1 - \frac{1}{4^2}) \cdot \ldots \cdot (1 - \frac{1}{20^2})$

$= (1 - \frac{1}{2}) \cdot (1 - \frac{1}{3}) \cdot (1 - \frac{1}{4}) \cdot \ldots \cdot (1 - \frac{1}{20}) \cdot (1 + \frac{1}{2}) \cdot (1 + \frac{1}{3}) \cdot (1 + \frac{1}{4}) \cdot \ldots \cdot (1 + \frac{1}{20})$

$= (\frac{1}{2})(\frac{2}{3})(\frac{3}{4}) \ldots (\frac{19}{20})(\frac{3}{2})(\frac{4}{3})(\frac{5}{4}) \ldots (\frac{21}{20})$ which reduces to $\frac{1}{20} \cdot \frac{21}{2} = \frac{21}{40}$

Thus a=21 and b = 40 so that b − a = 19. **Answer:** 19

Please note: Concepts used today will be repeated later this year.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 1995**

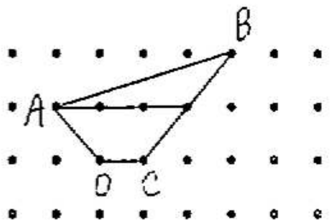
Solutions

F95J7. 100 cubes per face are painted in one color only. Thus there are 600 cubes painted in one color only. Generally, if each edge of a painted cube is cut into n ≅ segments, there are $6(n-2)^2$ cubes with one painted face. Answer: 600

F95J8. Let x = the number of grandchildren. In order for x – 2 to be divisible by 3, 4 and 5, x–2 must be divisible by the product $3 \cdot 4 \cdot 5 = 60$. The smallest such value of x–2 is 60 leading to x = 62. Answer: 62

F95J9. To get a terminal zero, the factor "2" must pair with a factor "5." Many more terms have 2 as a factor than 5, so concentrate on the 5's: The factors of 100! containing "5" are 5,10, 15,... 100. There are 20 of them. Note that 25, 50, 75 and 100 each have an extra factor of 5, giving 4 more factors. Thus there are twenty–four zeros in 100! Answer: 24

F95J10. Squaring both sides of $\sqrt{2+\sqrt{3}} = \sqrt{a} + \sqrt{b}$ gives $2 + \sqrt{3} = a+b+2\sqrt{ab}$ or $2+\sqrt{3} = a+b+\sqrt{4ab}$. Equating rational and irrational parts gives 2=a+b and 3= 4ab. Substitute a= 2–b in the second equation to get $4(2-b)b = 3$. This leads to the equation $4b^2 - 8b + 3 = 0$ which has roots $b = \frac{1}{2}, 3/2$. If $b = \frac{1}{2}$, $a = 3/2$. Reject this since we are given that a<b. If b = 3/2, $a = \frac{1}{2}$. Answer: $\left(\frac{1}{2}, \frac{3}{2}\right)$



F95J11. *Method I:* Connect points to form a triangle whose area is 1½ and a trapezoid with area 2. Adding the areas of the trapezoid and triangle gives the area of to be 3½. *Method II:* The polygon has 5 "lattice points" on its boundary and two in its interior. A polygon with "I" interior lattice points and "B" boundary lattice points (including the vertices) has area $A = \frac{1}{2}B + I - 1$. This is known as "Pick's Theorem." It also gives the area to be 3½. Answer: 3½

F95J12.

Page #s	# of 1's	# of 2's
1–9	1	1
10–19	11	1
20-29	1	11
30-99	7	7
100-109	11	1
110-119	21	1
120,121,122	4	4
Totals	56	26

Answer: 26

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1995**

Solutions

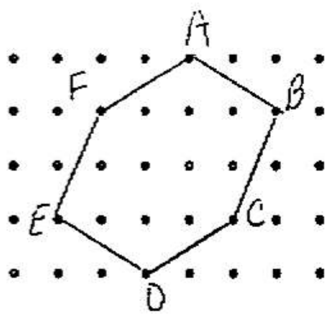
F95J13. 80 cubes in the "front" are painted in exactly two faces only. Include another 80 cubes in the "back." Each connecting edge from front to back has another 20 cubes painted in exactly two faces adding another 80 cubes for a grand total of 240 cubes. Generally, If each edge of a painted cube is cut into n segments, there are 12(n–2) cubes painted on two faces. Answer: 240

F95J14. $24! = 1\cdot3\cdot5\cdot\ldots\cdot23\cdot\underline{2\cdot4\cdot6\cdot\ldots\cdot24}$. The underlined part will help to compute the needed power of 2.

2^1 divides 2	2^1 divides 10	2^1 divides 18	The largest value of x so that
2^2 divides 4	2^2 divides 12	2^2 divides 20	2^x divides 24! is the sum of
2^1 divides 6	2^1 divides 14	2^1 divides 22	the exponents here, 22.
2^3 divides 8	2^4 divides 16	2^3 divides 24	Answer: 22

F95J15. Let x = the number of great–grandchildren. For x –3 to be divisible by 3, 10 and 14, it must be divisible by their least common multiple which is 210. Thus the smallest value of x–3 is 210 so that x = 213. Answer: 213

F95J16. The three numbers can be written in base 10 as 2a+2, 2a+6 and 3a +4. Setting up the following proportion from the geometric sequence:
$\frac{2a+6}{2a+2} = \frac{3a+4}{2a+6}$ leads to $4a^2 + 24a + 36 = 6a^2 + 14a + 8$ and $2a^2 - 10a - 28 = 0$ which has roots –2 and 7. Reject a= –2. Answer: 7



F95J17. The easiest method is to use Pick's Theorem (See F95J9.) The polygon has 6 "lattice points" on its boundary and ten in its interior. Using area $A = \frac{1}{2}B + I - 1$ with B = 6 and I = 10, the area is A = 3 + 10 – 1 = 12. Answer: 12

F95J18. Squaring $\sqrt{a+b} + \sqrt{b+c} = \sqrt{a+c}$ gives: $a+b+b+c + 2\sqrt{ab+ac+b^2+bc} = a+c$ which simplifies to $\sqrt{ab+ac+b^2+bc} = -b$. NOTE: b is negative! Square both sides again and simplify to get ab + ac + bc = 0. This leads to $a = \frac{-bc}{b+c}$. Experimenting with different values gives several answers, some of which are (6,–2,3), (3,–2,6), (5,–4,20), (20, –4, 5), (4,–3,12), (12,–3,4), (6,–5,30), and (30,–5,6). The smallest sum a+b+c is therefore 7. Answer: 7

January 10, 1996

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1995 NYCIML contests that you requested on the application form.

The following questions had different answers than the given one or were eliminated from the competitions.

	Question	Correct answer
Junior	F95J15 was eliminated	You cannot get a remainder of 3 when dividing by 3.
	F95J18 was eliminated	It should have read "a + b, b + c, and a + c are all greater than 0". Otherwise imaginary numbers could be considered.
Senior A	F95S14	3
	F95S24	$162\pi + 81\sqrt{3}$
	F95S25 was eliminated	It should have read " a and b are rational". The correct answer would then be (3, 1). Otherwise there are infinite solutions.

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML