NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE

PART I: TIME: 10 MINUTES SPRING, 1995

S95S1

Compute the value of the smallest positive integer with exactly twelve divisors.

S95S2

A right triangle has legs of length a and b. An equilateral triangle is drawn on the side of length a and a square is drawn on the side of length b. A regular hexagon is drawn on the hypotenuse. Let A, B, and C be the areas of the triangle, square, and hexagon respectively. Write an equation expressing C explicitly in terms of A and B.

PART II: TIME: 10 MINUTES SPRING, 1995

S95S3

Let $f(n) = n^2 + n + 17$. Compute the smallest positive integer value of n so that f(n) is a perfect square.

S95S4

Ten cups of fluid A are in container A and ten cups of fluid B are in container B. One cup of A is taken from container A and added to container B. Then one cup of the saturated mixture in B is taken and added to container A. Compute the absolute value of the difference between the percent of fluid A now in container A and the percent of fluid B now in container B.

PART III: TIME: 10 MINUTES SPRING, 1995

S95S5

A circle with radius a is tangent to line L. Two congruent circles of radius b are each tangent to line L, the first circle, and to each other. If the length on line L between the points of tangency of the latter circles is 20, compute the value of a.

S95S6

The area of a square is 1/2 and is divided into three regions of equal area by two lines parallel to a diagonal. Compute the distance between these lines.

ANSWERS

1. 60
2. $C = 6A + (3\sqrt{3}/2)B$
3. 16
4. 0
5. 2.5
6. $(3 - \sqrt{6})/3 = 1 - (\sqrt{6}/3)$

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO

PART I: TIME: 10 MINUTES SPRING, 1995

S95S7

Square ABCD is inscribed in circle O. Square PQRS is inscribed in semicircle ABC so that two vertices lie on the diameter AOC. Compute the ratio of the area of square PQRS to the area of square ABCD.

S95S8

A young girl's father is eleven times as old as she is. Her grandfather is as old now as the father will be when the girl is one-third the grandfather's age then. If the girl is g years old now, express the grandfather's present age in terms of g.

PART II: TIME: 10 MINUTES SPRING, 1995

S95S9

If $(x + 1/x)^2 = 3$, compute the value of $x^3 + 1/x^3$.

S95S10

Compute the length of the longest string of consecutive 0's in the base 2 representation of the base 8 numeral 50403.

PART III: TIME: 10 MINUTES SPRING, 1995

S95S11

The numbers a^2, b^2, and c^2 form an arithmetic progression. If a + b = 2 and a + c = 5,

compute $\dfrac{c-a}{b-a}$.

S95S12

Point P is in the exterior of rectangle ABCD so that PA = 1 and PD = 7. If PC is twice as long as PB, compute the value of PB.

<u>ANSWERS</u>

7. 2/5
8. 19g
9. 0
10. 6
11. 4/5
12. 4

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE

PART I: TIME: 10 MINUTES SPRING, 1995

S95S13

If the equation $x^3 + \sqrt{2}\, x^2 - \sqrt{5}\, x + k = 0$ has two distinct real roots that are additive inverses of each other, compute the value of k.

S95S14

Two congruent circles of radius 6 with distinct centers lie within a rectangle so that exactly three sides of the rectangle are tangent to each circle. If the dimensions of the rectangle are 12 and 20, compute the area of the closed convex polygon formed by the centers of the circles and the points of intersection of the circles.

PART II: TIME: 10 MINUTES SPRING, 1995

S95S15 If $x + y = a$ and $xy = b^2$, express in simplest form $(\sqrt{x} + \sqrt{y})^4$ in terms of a and b.

S95S16

Two congruent circles with centers O and P and radius $\sqrt{3}$ intersect so that the center each circle lies on the other circle. Let A and B be the points of intersection. BC is a chord containing P and BD is a chord containing O. Compute the area of triangle CBD.

PART III: TIME: 10 MINUTES SPRING, 1995

S95S17 Find the sum of the infinite series
$1/10 + 2/10^2 + 3/10^3 + 4/10^4 + \ldots$.

S95S18 Compute $\sin 10° \sin 50° \sin 70°$.

<u>ANSWERS</u>

13. $-\sqrt{10}$
14. $16\sqrt{5}$
15. $a^2 + 4ab + 4b^2$
16. $3\sqrt{3}$
17. 10/81
18. 1/8

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR

PART I: TIME: 10 MINUTES SPRING, 1995

S95S19

On each side of an equilateral triangle of side length 1 a square is drawn. The vertices of neighboring squares are joined by segments so that these segments along with a side from each square form a closed convex hexagon. Compute the area of the hexagon.

S95S20

A math student started a problem at a time between 4 pm and 5 pm when the hour hand and minute hand of the clock were coincident. The student finished less than an hour later when the hands formed a straight angle. Compute the exact time, in minutes, that the student used to complete the problem.

PART II: TIME: 10 MINUTES SPRING, 1995

S95S21

The measure of the interior angle of a regular polygon with M sides is 2/3 the measure of the interior angle of a regular polygon with N sides. Find all ordered pairs (M, N).

S95S22

Compute the least number of terms necessary so that the finite partial sum of the alternating series

$$1 - .8 + 1.5 - 1.2 + 2.0 - 1.6 + 2.5 - 2.0 + \dots$$

is greater than 100.

PART III: TIME: 10 MINUTES SPRING, 1995

S95S23

The nine non zero digits are written on a sheet of paper. One digit was crossed out. The remaining eight were separated into 3 sets of at least two digits in each. If the sums of the elements in each set are 8, 11, and 18, which of the nine non zero digits was crossed out?

S95S24

Let A be the area between the graph of $y = \log[x]$ and the x-axis for $2 \le x < 10$. ([x] represents the greatest integer $\le x$.) Compute A if $\log 2 = .3$, $\log 3 = .5$, $\log 5 = .7$, and $\log 7 = .8$.

ANSWERS

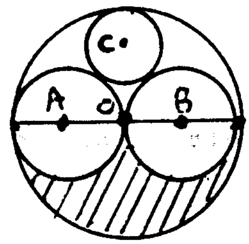
19. $3+\sqrt{3}$
20. 32 and 8/11 minutes
21. (3, 4), (4, 8), (5, 20)
22. 79
23. 8
24. 5.6

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE

PART I: TIME: 10 MINUTES SPRING, 1995

S95S25 Three fair dice are rolled simultaneously. Find the probability that the three numbers obtained can be arranged in an arithmetic progression.

S95S26 Circles A and B are tangent to each other and to Circle O. A and B are also on the diameter of O. Circle C is tangent to these three circles. If the area of the shaded region (called an arbelos) is 4π, compute the radius of Circle C.



PART II: TIME: 10 MINUTES SPRING, 1995

S95S27 Two poles are 20 feet and 80 feet tall respectively. Their bases are 100 feet from each other. If cables join the top of each pole to the base of the other, how high is the point of intersection of the cables?

S95S28 Let the sequence $\{a_n\}$ be generated by $a_n = [\sqrt{n}]$ for integers n = 0,1,2,3,... ([x] represents the greatest integer $\le$ x) Compute the number of terms in this sequence that are even and less than 50.

PART III: TIME: 10 MINUTES SPRING, 1995

S95S29 An isosceles triangle has sides 8, 10, and 10. Find the length of the radius of the circumscribed circle.

S95S30 Let S be the set of positive integers consisting of 5 distinct non zero digits. Let x be an element of S. Let d(x) be the positive difference of x and the number obtained by reversing the digits of x. Compute the largest value of x so that the sum of the digits of d(x) is 18.

ANSWERS

25. 16
26. 4/3
27. 1/6
28. 1225
29. $\frac{25\sqrt{21}}{21}$
30. 97685

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION A - SOLUTIONS

CONTEST NUMBER ONE - SPRING, 1995

S95S1
The prime factorization of the integer $p_1^{a1} \times p_2^{a2} \times p_3^{a3} \times \ldots$ must satisfy $(a1+1)(a2+1)(a3+1)\ldots = 12$. Since we want the smallest such integer, the possibilities are 2^{11}, $2^5 \times 3$, $2^3 \times 3^2$, $2^2 \times 3 \times 5$. The last is the smallest, 60.

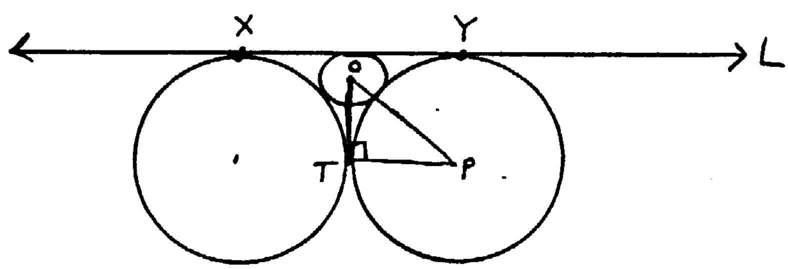
S95S2

The area of the hexagon, $C = 6(\sqrt{3}/4)c^2 = (3\sqrt{3}/2)(a^2 + b^2)$. The area of the equilateral triangle is $A = (\sqrt{3}/4)a^2$ and the area of the square is $B = b^2$. Therefore, $C = (3\sqrt{3}/2)(4A/\sqrt{3} + B) = 6A + (3\sqrt{3}/2)B$.

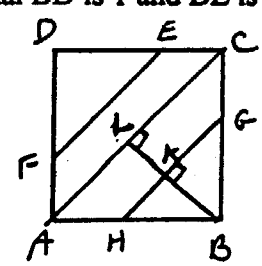
S95S3
$f(n) = n(n+1) + 17$. If $17 = n + 1$, then $f(n) = (n+1)^2$. Therefore, $n=16$. Quick investigation shows that this is the smallest such positive integer producing a perfect square.

S95S4
After transferring the first cup, container A has 9 cups of fluid A and the mixture in container B is 1/11 part fluid A and 10/11 part fluid B. Transferring the second cup yields 9 + 1/11 cups of fluid A in container A and 10/11 x 10 = 100/11 cups of fluid B in container B. These amounts are equal. Therefore, the difference of the percentages is 0.

S95S5
Let X and Y be the points of tangency of the latter circles to line L, T be their common point, and O and P be the centers of the first circle and one of the latter circles respectively. Triangle OTP is a right triangle and we have $OT^2 + PT^2 = OP^2$. $PT = b = (½)XY = 10$, $OT = b - a = 10 - a$,
and $OP = b + a = 10 + a$. Thus, $10^2 + (10 - a)^2 = (10 + a)^2$ giving
$a^2 - 20a + 200 = a^2 + 20a + 100$ and $a = 2.5$.

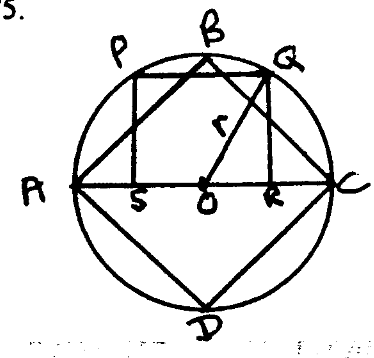


S95S6
Let the square be labeled ABCD as shown and EF and GH be the dividing lines. Also, let K and L be the intersection points of diagonal BD with GH and CA respectively. The distance we seek is 2KL.
The ratio of the area of triangle GHB to the area of triangle CAB is 2/3. Therefore, the ratio of altitude BK to altitude BL is $\sqrt{2}/\sqrt{3}$.
Since the area of ABCD is 1/2, a side of the square is $\sqrt{2}/2$. Therefore, diagonal BD is 1 and BL is 1/2.
$2KH = 2(BL - BK) = 2(1/2 - (\sqrt{2}/\sqrt{3})(1/2)) = 1 - \sqrt{2}/\sqrt{3} = (3 - \sqrt{6})/3$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION A - SOLUTIONS
CONTEST NUMBER TWO - SPRING, 1995

S95S7
Let PQRS be the square inscribed in the semicircle as shown. Let r be the radius of circle O. The area of ABCD is $\frac{1}{2}(2r)^2 = 2r^2$. $OR = \frac{1}{2}SR = \frac{1}{2}QR$ and $QR^2 + OR^2 = OQ^2 = r^2$ which yields the area of $PQRS = QR^2 = 4r^2/5$. Therefore the ratio we seek is 2/5.



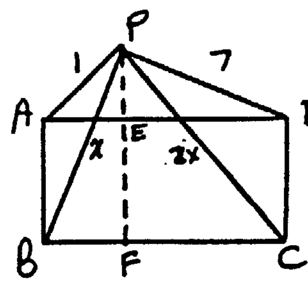
S95S8
Let x be the grandfather's present age, f be the father's present age, and n be the number of year's from now when the girl is 1/3 the grandfather's age. We have $x = f + n$, $f = 11g$, and
$g + n = (x + n)/3 = (f + n + n)/3 = (11g + 2n)/3$. This yields $n = 8g$ and, therefore, $x = 11g + 8g = 19g$.

S95S9
$(x + 1/x)^2 = 3,\ x^2 + 2 + 1/x^2 = 3.$
$x^2 + 1/x^2 = 1,\ x + 1/x = \sqrt{3}.$
$(x^2 + 1/x^2)(x + 1/x) = \sqrt{3}.$
$x^3 + x + 1/x + 1/x^3 = \sqrt{3}.$
$x^3 + \sqrt{3} + 1/x^3 = \sqrt{3}.$
$x^3 + 1/x^3 = 0.$
(Note: Solution is the same if $x + 1/x = -\sqrt{3}$.

S95S10
Every digit in the base 8 numeral 50403 represents the base ten equivalent of a triplet in the base 2 representation. That is 5 is 101, 0 is 000, 4 is 100, and 3 is 011 giving the base 2 numeral 101000100000011. The length of the longest string of consecutive 0's is 6.

S95S11
Since the numbers are in an arithmetic progression, $a^2 = b^2 - d$ and
$c^2 = b^2 + d$ where d is the common difference between terms. Therefore, $c^2 - a^2 = 2d$ or $c^2 - a^2 = 2(b^2 - a^2)$. This gives $(c^2 - a^2)/(b^2 - a^2) = 2$. Factoring and using the given values for a + b and a + c, we have $(c-a)/(b-a) = 4/5$.

S95S12
Let PB = x and PC = 2x.
From P draw PEF perpendicular to AD and BC. Using the four right triangles AEP, BFP, DEP, and CFP we derive $PF^2 - PE^2 = 4x^2 - 49 = x^2 - 1$. Solving yields x = 4.



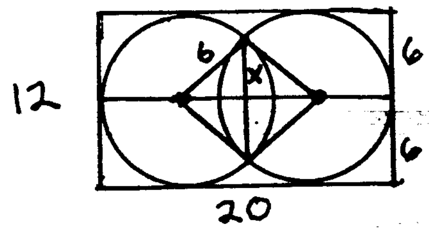
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION A - SOLUTIONS
CONTEST NUMBER THREE - SPRING, 1995

S95S13

Suppose the roots are a and -a, a>0. Then we have the two equations:
$a^3 + \sqrt{2}\,a^2 - \sqrt{5}\,a + k = 0$ and $-a^3 + \sqrt{2}\,a^2 + \sqrt{5}\,a + k = 0$. Adding the two equations yields $2\sqrt{2}\,a^2 + 2k = 0$ and $k = -\sqrt{2}\,a^2$. Subtracting the two equations yields $2a^3 - 2\sqrt{5}\,a = 0$ giving $a^2 = \sqrt{5}$. Therefore, $k = -\sqrt{10}$.

S95S14

The radii of the circles are each 6 and the distance between the centers is 8. The polygon is a rhombus and its area is 4(½)(4)(x) where x is one-half the length of the common chord. Clearly, $x^2 = 6^2 - 4^2 = 20$ giving $x = 2\sqrt{5}$. The area is $16\sqrt{5}$.

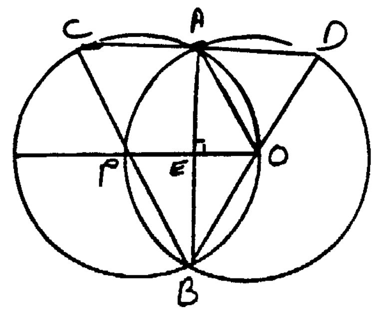


S95S15

$$(\sqrt{x} + \sqrt{y}\,)^4 = \sqrt{x}^{\,4} + 4\sqrt{x}^{\,3}\sqrt{y} + 6\sqrt{x}^{\,2}\sqrt{y}^{\,2} + 4\sqrt{x}\,\sqrt{y}^{\,3} + \sqrt{y}^{\,4}$$
$$= x^2 + 6xy + y^2 + 4(x + y)\sqrt{xy}$$
$$= (x + y)^2 + 4xy + 4(x + y)\sqrt{xy} = a^2 + 4b^2 + 4ab = (a + 2b)^2 = a^2 + 4ab + 4b^2$$

S95S16

The area of the triangle is ½(CD)(AB). The common chord AB is bisected by the diameter of either circle and creates segments of lengths (3/2)r and (1/2)r on the diameter. Letting each half of AB be x, we have $x^2 = (3/4)r^2$ giving $AB = \sqrt{3}\,r$. Clearly, triangle AEO is a 30-60-90 right triangle. Therefore, angle OAD is 60, triangle OAD is equilateral, and AD = r. CD = 2r and the area of triangle CBD = $½(2r)(\sqrt{3}\,r) = \sqrt{3}\,r^2$. Since $r = \sqrt{3}$, the area is $3\sqrt{3}$.



S95S17 The series is the sum of other series:
$(1/10 + 1/10^2 + 1/10^3 + \ldots) + (1/10^2 + 1/10^3 + 1/10^4 + \ldots) + (1/10^3 + 1/10^4 + 1/10^5 + \ldots) +$ etc. Using Sum $= a_1/(1 - r)$, these series have sums of $1/9 + 1/90 + 1/900 + \ldots = (1/9)/(1 - 1/10) = 10/81$.

S95S18

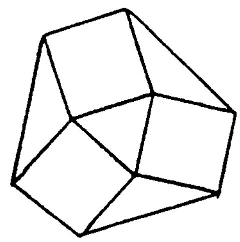
Using sinAsinB = (½)(cos(A-B) - cos(A+B)), we have

$$\sin 10° \sin 50° \sin 70° = (½)\sin 50° (\cos 60° - \cos 80°)$$
$$= (¼)\sin 50° - (½)\sin 50° \cos 80°$$
$$= (¼)\sin 50° - (½)\sin 50° \sin 10°$$
$$= (¼)(\sin 50° - 2\sin 50° \sin 10°)$$
$$= (¼)(\sin 50° - (\cos 40° - \cos 60°))$$
$$= (¼)(\sin 50° - \sin 50° + ½) = 1/8.$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION A - SOLUTIONS
CONTEST NUMBER FOUR - SPRING, 1995

S95S19

Clearly the area of the inner triangle is $\sqrt{3}/4$ and the area of each of the three squares is 1. Each of the other three triangles has area equal to $(\frac{1}{2})\sin 120° = \sqrt{3}/4$. Therefore, the total area is $\sqrt{3}/4 + 3 + 3\sqrt{3}/4 = 3 + \sqrt{3}$.



S95S20

Consider the minutes as marking positions on the circle. Let a represent the position of the minute hand and b be equal to the number of hours x 5. In our problem, $b = 4 \times 5 = 20$. At a minutes past the hour, the hour hand is at the position $b + a/12$. The problem leads to us starting at a position satisfying $20 + a/12 = a$. Therefore, the starting time has the minute hand at $a = 20(12/11)$. The ending position must satisfy $20 + a/12 = a - 30$. Therefore, the ending time has the minute hand at $a = 50(12/11)$. The difference in time is $30(12/11) = 30 + 30/11 = 32$ and $8/11$ minutes

S95S21 $(N - 2)\cdot 180/N = (3/2)\cdot(M - 2)\cdot 180/M$. $N = 4M/(6 - M)$. The only M's that will produce positive integers for N are 3, 4, and 5. Thus, (3,4), (4,8) and (5,20) are all the solutions required.

S95S22 k = 1 3 5 7

$1 \underbrace{- .8 + 1.5}_{+.7} \underbrace{- 1.2 + 2.0}_{+.8} \underbrace{- 1.6 + 2.5}_{+.9} - 2.0 + \ldots$

Sum = 1 +.7 +.8 +.9 + ...

We get above 100 for the first time when we add a number. By grouping the terms of the series pairwise we have that the sum is $1 + .7 + .8 + .9 + \ldots$ We want $.7 + .8 + .9 + \ldots + a_n > 99$ for some integer n. Then $a_n = .7 + (n-1)(.1)$ and $S_n = (n/2)(1.4 + (n - 1)(.1)) > 99$. This gives the inequality $n^2 + 13n - 1980 > 0$. Therefore, n is between 38 and 39. Therefore, $n = 39$ is the first such integer. Therefore, there must be $k = 2(39) + 1 = 79$ terms in order to be greater than 100.

S95S23

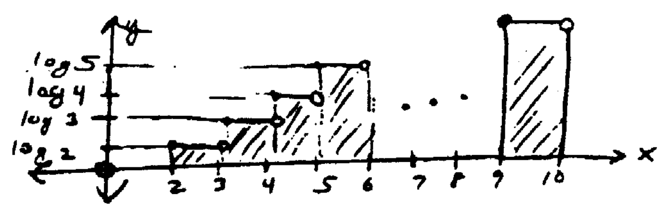
The way in which the digits were separated is irrelevant. Since the sum of all 9 non zero digits is 45 and the total sum of the remaining 8 digits is 37, the deleted digit is 8.

S95S24

The graph is a step function with range {log2, log 3, ... , log9}. The area under each portion of the graph and above the x-axis is a rectangle whose width is 1. Therefore,

$A = \log 2 + \log 3 + \ldots + \log 9 = \log(9!) = \log(2^7 \times 3^4 \times 5 \times 7)$

$= 7\log 2 + 4\log 3 + \log 5 + \log 7 = 7(.3) + 4(.5) + .7 + .8 = 5.6.$

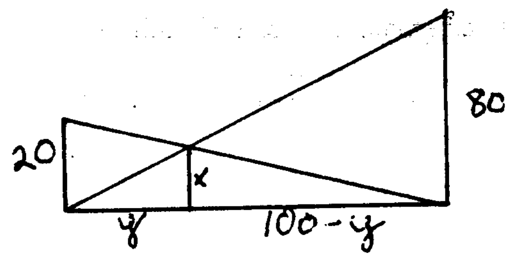


NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION A - SOLUTIONS
CONTEST NUMBER FIVE - SPRING, 1995

S95S25 There are 6^3 ways the dice can land. There are 6 different arithmetic progressions possible: 1,2,3; 2,3,4; 3,4,5; 4,5,6; 1,3,5; and 2,4,6. For each of these, there are 3! or 6 ways they can turn up on the three dice. $6 \cdot 6/6^3 = 1/6$.

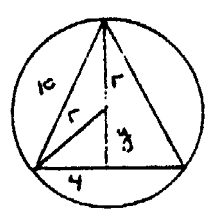
S95S26
Let r be the radius of circles A and B. Therefore, 2r is the radius of circle O. The area of the arbelos is equal to $(\frac{1}{2})\pi(2r)^2 - 2(\frac{1}{2})\pi r^2 = \pi r^2$. Since this is equal to 4π, $r = 2$. Let x be the radius of circle C. In right triangle COB, we have $OC = 2r - x$, $OB = r$, and $BC = r + x$. Thus,
$(2r - x)^2 + r^2 = (r + x)^2$. We find that $x = 2r/3 = 4/3$.

S95S27 $x/80 = y/100$ or $100x = 80y$.
$x/20 = (100 - y)/100$ or
$100x = 2000 - 20y = 80y$
$y = 20$ and $x = 16$.



S95S28
The sequence repeats integers from $n = k^2$ to $n = (k + 1)^2 - 1$. Therefore, there are strings of k of length $2k + 1$.
That is, 0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4...
Therefore, there are $1+5+9+13+...+(2\times48 + 1)$
$= (4\times0 + 1) + (4\times1 + 1) + (4\times2 + 1) + ... + (4\times24 + 1)$
$= 4(0 + 1 + 2 + ... + 24) + 25 = 4(24)(25)/2 + 25 = 1225$ even integers before 50 occurs.

S95S29 The height $h = \sqrt{84}$.
$r^2 = 4^2 + y^2 = 4^2 + (h-r)^2$
$= 16 + h^2 - 2hr + r^2 = r^2$
$r = (16 + h^2)/2h = (16 + 84)/2\sqrt{84}$
$= 50/\sqrt{84} = 25\sqrt{21}/21$.



S95S30
It is easily proven using modulo 9 arithmetic that the sum of the digits of the difference must be a multiple of 9. Examination of the situation reveals that the digit in the middle of the difference will be 0 or 9.
Consider abcde - edcba. (Clearly, a > e). Suppose the middle digit is 9. This means that we needed to "borrow" 1 from c (d<b) and 1 from b. The sum of the digits of the difference from right to left is
(e+10-a)+(d-1+10-b)+9+(b-1-d)+(a-e) = 27. Therefore, the integer we seek must not produce a "borrowing" situation. Since we are looking for the largest such integer, we investigate 9xxxx. It is impossible for our integer to be of the forms 98xxx or 978xx. Immediately following is an acceptable integer, 97685.

May 20, 1995

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1995 NYCIML contests that you requested on the application form.

The following are the corrected or alternative answers for the enclosed contests.. The original answers were inaccurate.

	Question	Correct answer
Senior A	S95S16	$3\sqrt{3}$ or $\frac{3\sqrt{3}}{2}$ if C=P or $\frac{3\sqrt{3}}{4}$ if D= P
	S95S25	1/6 or 7/36 (if d = 0)
	S95S27	16
Senior B	S95B28	1/6 or 7/36 (if d = 0)
Junior	S95J12	$\frac{\sqrt{3}}{4}$ or $\frac{\sqrt{3}}{16}$ (if the process was repeated an extra time)

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML

~~only one~~