NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE

PART I: TIME: 10 MINUTES SPRING, 1994

S94S1 Find the smallest number that 3600 must be multiplied by to form a perfect cube.

S94S2 If A, B, C are positive integers less than ten then find all such triples (A,B,C) that satisfy $4!A + 5!B + 6!C = 1992$.

PART II: TIME: 10 MINUTES SPRING, 1994

S94S3 Three fair dice are tossed. Compute the probability that the three numbers that appear on the top face can be arranged to form an arithmetic sequence with a common difference of one.

S94S4 Compute $\sin^4 10^\circ + \sin^4 35^\circ + \sin^4 55^\circ + \sin^4 80^\circ$

PART III: TIME: 10 MINUTES SPRING, 1994

S94S5 How many more positive integral factors does 48^M have than 32^M? Express your answer in terms of M, a natural number?

S94S6 In a sequence with initial term T_0 satisfying

1. $T_N = T_{N-1} + 1/N(N+1)$ for $N \geq 1$
2. $T_{101} = 107/102$.

Find T_0.

ANSWERS

1. 60
2. (3,4,2) (8,9,1) (8,3,2)
3. 1/9
4. 3/2
5. $4M^2$
6. 1/17

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER TWO

PART I: TIME: 10 MINUTES

SPRING, 1994

S94S7 Compute the area between $x^2 + y^2 = 9$ and $|x| + |y| = 3$.

S94S8 The polynomial $k^3x^2 + 14kx + 24$ is divisible by $(x+2)$ and $(x+T)$. Find all real numbers T could be.

PART II: TIME: 10 MINUTES

SPRING, 1994

S94S9 Consider the sequence of pentagonal numbers 1, 5, 12, 22, 35, ... Find the 50^{th} pentagonal number.

S94S10 Triangle ABC has integral sides which are in arithmetic progression. Find the minimum perimeter of such a triangle if the cosine of angle ABC is -1/8.

PART III: TIME: 10 MINUTES

SPRING, 1994

S94S11 Given the identity $a/(10^x + 2) + b/(10^x - 1) = (4 \cdot 10^x + 2)/(10^x + 2)(10^x - 1)$ for positive rational values of x, compute $a \cdot b$.

S94S12 Let $f(x) = (x^3 - 331)/(x - 11)$. Suppose p is chosen randomly from the set (1,2,3, ..., 1100), then what is the probability that f(p) is an integer?

ANSWERS

7. $9\pi-18$
8. -4/9, 3/2, 12
9. 3725
10. 51
11. 4
12. 1/50

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER THREE

PART I: TIME: 10 MINUTES SPRING, 1994

S94S13 A family has three children. The product of their ages is 648, and one child's age is the cube of another child's age. Compute the sum of the children's ages if all ages are whole numbers of years.

S94S14 Compute in degrees:
$\text{Tan}^{-1}(1/2) + \text{Tan}^{-1}(1/5) + \text{Tan}^{-1}(1/8)$.

PART II: TIME: 10 MINUTES SPRING, 1994

S94S15 The sum of the squares of the roots of the equation $x^2 + 4hx = 5$ is 154. Compute the $|h|$.

S94S16 How many six digit integers have their digits in increasing order (with no repetition of digits)?

PART III: TIME: 10 MINUTES SPRING, 1994

S94S17 Evaluate the sum
$1/(2\cdot3) + 1/(3\cdot4) + 1/(4\cdot5) + \ldots + 1/(256\cdot257)$.

S94S18 Regular hexagon ABCDEF is circumscribed around a circle of radius $2 + \sqrt{3}$. Line M is tangent to the circle and parallel to AC. What is the length of that portion of M which is contained in the hexagon?

ANSWERS

13. 38
14. 45°
15. 3
16. 84
17. 255/514
18. 2

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR

PART I: TIME: 10 MINUTES SPRING, 1994

S94S19 Consider the points A(-2,-3) and B(5,2) in the rectangular coordinate system. Compute the value of k such that the point C(2,k) minimizes the sum AC + CB.

S94S20 What is the largest integer that leaves the same remainder when it divides 1992, 5556, or 12090?

PART II: TIME: 10 MINUTES SPRING, 1994

S94S21 In the expression (1/x + 1/y)/(1/x - 1/y), x and y are positive integers less than or equal to 200 and x does not equal y. Compute the largest value of (1/x + 1/y)/(1/x - 1/y).

S94S22 Find all integer pairs (a,b) which satisfy $(a - 8)(a - 10) = 2^b$.

PART III: TIME: 10 MINUTES SPRING, 1994

S94S23 x ounces of a salt solution is x% salt. y ounces of water are added to produce an (x - 30)% solution. Compute y in terms of x.

S94S24 Noting that 63·14 = 882, solve for all real x: $(x^2 - 14x + 42)^3 = 63x^2 - 882x + 2808$.

ANSWERS

19. -1/7
20. 594
21. 399
22. (6,3), (12,3)
23. y = 30x/(x - 30)
24. 3,5,6,8,9,11

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE

PART I: TIME: 10 MINUTES SPRING, 1994

S94S25 If $g((x - 4)/3) = x$, compute $g(x)$.

S94S26 Find xy if $x + y + \sqrt{xy} = 45$ and $x^2 + y^2 + xy = 1992$.

PART II: TIME: 10 MINUTES SPRING, 1994

S94S27 Solve the following for x: $\log_a(\log_b x) = c$.

S94S28 The perimeter of a triangle is 48 and one side is 21. If the sides and the area of this triangle are all integers, then compute the length of the shortest side.

PART III: TIME: 10 MINUTES SPRING, 1994

S94S29 Compute the number of ordered pairs (a,b), where a and b are positive integers, that satisfy $ab/(a + b) = 3$.

S94S30 Compute the value of $(\cos 10^\circ - \sqrt{3}/4)^2 + \cos^2 20^\circ$.

ANSWERS

25. $g(x) = 3x + 4$
26. 121/900
27. b^{a^c}
28. 10
29. 3
30. 19/16

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER ONE SPRING, 1994

S94S1 $3600 = 2^4 \cdot 3^2 \cdot 5^2$. To have a perfect cube the exponents must be multiples of 3. Thus, the smallest number that we can multiply 3600 by to form a perfect cube is $2^3 \cdot 3 \cdot 5$ or 60.

S94S2 We have $A + 5B + 30C = 83$, thus $A \equiv 3 \pmod 5$ or $A = 3$ or 8. If $A = 3$ then $B + 6C = 16$ or $C = 2$ and $B = 4$. If $A = 8$ then $B + 6C = 15$ and $(C,B) = (1,9)$ or $(2,3)$. Hence the answers are $(3,4,2)$, $(8,9,1)$, and $(8,3,2)$.

S94S3 The outcomes must be 1,2,3 or 2,3,4 or 3,4,5 or 4,5,6. The probability is therefore $4 \cdot 6/6^3 = 1/9$.

S94S4 Call the sum S.

$S = \sin^4 10^\circ + \cos^4 55^\circ + \sin^4 55^\circ + \cos^4 10^\circ$

$S = (\sin^2 10^\circ + \cos^2 10^\circ)^2 - 2\sin^2 10^\circ \cos^2 10^\circ + (\sin^2 55^\circ + \cos^2 55^\circ)^2 - 2\sin^2 55^\circ \cos^2 55^\circ$

$2S = 4 - \sin^2 20^\circ - \sin^2 110^\circ$

$2S = 4 - \sin^2 20^\circ - \sin^2 70^\circ$

$2S = 4 - \sin^2 20^\circ - \cos^2 20^\circ$

$2S = 3$ or $S = 3/2$

S94S5 Writing 48^M and 32^M in prime factorization form, we obtain $48^M = (2^4 \cdot 3)^M = 2^{4M} \cdot 3^M$ and $32^M = (2^5)^M = 2^{5M}$. Thus, 48^M has $(4M+1)(M+1) = 4M^2 + 5M + 1$ factors and 32^M has $5M+1$ factors. So, 48^M has $4M^2$ more factors than 32^M.

S94S6 $T_N - T_{N-1} = 1/N(N+1)$.

Now $\sum_{K=1}^{N} (T_K - T_{K-1}) = \sum_{K=1}^{N} 1/K(K+1)$

$T_N - T_0 = \sum_{K=1}^{N} (1/K - 1/(K+1)) =$

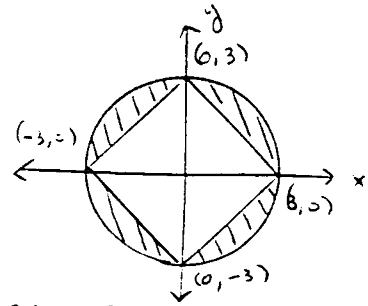
$(1/N - 1/(N+1)) + (1/(N-1) - 1/N) + \ldots + (1 - 1/2)$

$T_N - T_0 = 1 - 1/(N+1)$. Now $T_{101} - T_0 = 1 - 1/102$.

$107/102 = T_0 + 101/102$ or $T_0 = 1/17$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER TWO SPRING, 1994

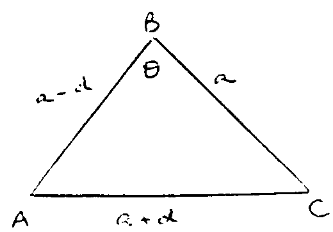
S94S7 The graphs of the two given equations are shown in the figure. The area of the circle is 9π and the area of the square is 18. Thus, the required area is $9\pi-18$.



S94S8 $x = -2$ is a root and therefore $4k^3 - 28k + 24 = 0$
$k^3 - 7k + 6 = (k-1)(k^2 + k - 6) = 0$
$(k-1)(k-2)(k+3) = 0$ or $k = 1, 2,$ or -3. Since -2 and $-T$ are roots we have $(-2)(-T) = 2T = 24/k^3$ or $T = 12/k^3$.
$T = 12, 3/2,$ or $-4/9$.

S94S9 Note that the successive differences between consecutive terms form the sequence 4,7,10,13,...148. The sum of this arithmetic sequence is $49\cdot(4+148)/2 = 3724$. Adding in the 1 we obtain $3724 + 1 = 3725$ as the 50^{th} term.

S94S10 Let $\cos\theta = -1/8$ and let a be an integer then we have the following situation with d an integer. Since $\theta > 90^{\circ}$ we have $AC>AB$ and $AC>BC$. Now
$-1/8 = (a^2 + (a-d)^2 - (a+d)^2)/2a(a-d)$
$-1/8 = (a - 4d)/2(a - d)$ or $5a = 17d$. The perimeter is $3a$, thus let $a=17$ and the answer is 51.



S94S11 Since we have an identity, we conclude
$a(10^x - 1) + b(10^x + 2) = 4\cdot 10^x + 2$.
Let $x=0$, to obtain $3b=6$ or $b=2$.
Let $x=1$, to obtain $9a + 12b = 42$ or $a=2$. Thus, $a\cdot b=2\cdot 2=4$.

S94S12 $f(x) = x^2 + 11x + 121 + 1000/(x-11)$. Thus if $(p-11)|1000$ then $f(p)$ is an integer. Now $1000 = 2^3\cdot 5^3$ has $(3+1)(3+1) = 16$ divisors. Now if $p-11 = -1, -2, -4, -5, -8, -10$ then $f(p)$ in an integer. If $p-11$ = any of the 16 divisors then $f(p) \in Z$. Thus, there are 22 such p in the set and the probability is $22/1100 = 1/50$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER THREE SPRING, 1994

S94S13 $648 = 2^3 \cdot 3^4$. Since one child's age is the cube of another, the ages are 2^3, 3, and 3^3. The sum is therefore, $8 + 3 + 27 = 38$.

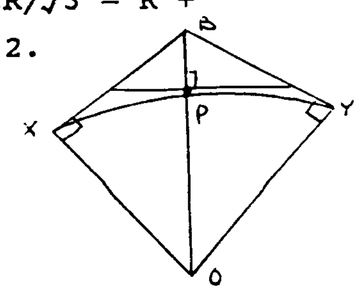
S94S14 Let $A = \text{Tan}^{-1}(1/2)$, $B = \text{Tan}^{-1}(1/5)$, $C = \text{Tan}^{-1}(1/8)$, and $\tan\theta$ be represented as T_θ. Now $T_{A+B+C} = (T_{A+C} + T_C)/(1 - T_{A+B} \cdot T_C)$. $T_{A+B} = (T_A + T_B)/(1 - T_A T_B) = (1/2 + 1/5)/(1 - (1/2)(1/5)) = 7/9$. Hence $T_{A+B+C} = (7/9 + 1/8)/(1 - (7/9)(1/8)) = 1$. Thus, $A+B+C = 45^\circ$.

S94S15 We know the sum of the roots, r and s, is $r + s = -4h$ and the product of the roots is $rs = -5$. $(r+s)^2 = r^2 + s^2 + 2rs$ or $r^2 + s^2 = (r+s)^2 - 2rs = (-4h)^2 - 2(-5) = 16h^2 + 10$. Thus, $16h^2 + 10 = 154$ or $h^2 = 9$ and $|h| = 3$.

S94S16 Pick six digits (w/o repitition) from {1, ..., 9}. Remember zero can't be the left most digit. Thus, the answer is ${}_9C_6 = 84$.

S94S17 The given sum can be rewritten as $(1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5) + \ldots + (1/256 - 1/257)$. This telescoping series yields $(1/2 - 1/257) = 255/514$.

S94S18 Let X, Y be two points of tangency on AB and BC, and let O be the center. MN is the sought line segment and P is the intersection of MN and OB. Now angle $XBY=120^\circ$, angle X = angle Y = angle BPN = 90°, and angle $XOY=60^\circ$. Let MN = L and OX=R. Angle $XOB=30^\circ$ and $OB = R\sec 30^\circ = 2R/\sqrt{3}$. Angle BNP $= (180^\circ-120^\circ)/2 = 30^\circ$ and $BP = (PN)\tan 30^\circ = (L/2)(1/\sqrt{3}) = L/2\sqrt{3}$. Now $OB = OP + PB$ and $2R/\sqrt{3} = R + L/2\sqrt{3}$ and $L = 2(2 - \sqrt{3})R = 2(2 - \sqrt{3})(2 + \sqrt{3}) = 2$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER FOUR SPRING, 1994

S94S19 The minimum value of AC + CB occurs when all three points lie on the same line. Thus, consider the slopes: $(k-2)/(2-5) = (2- (-3))/(5 - (-2))$ or $k = -1/7$.

S94S20 Let the answer be k, then $1992 = kP_1 + R$, $5556 = kP_2 + R$, and $12090 = kP_3 + R$, where P_1, P_2, P_3 are integers with $0 \le R \le k$. Now subtract and $k(P_2 - P_1) = 3564$ and $k(P_3 - P_2) = 6534$. We know that $k | 2^2 \cdot 3^4 \cdot 11$ and $k | 2 \cdot 3^3 \cdot 11^2$. The largest such k is $2 \cdot 3^3 \cdot 11 = 594$.

S94S21 $(1/x + 1/y)/(1/x - 1/y) = (y + x)/(y - x)$. In order to maximize $(y + x)/(y - x)$, choose x and y such that the numerator is the largest and the denominator is the smallest. Thus, choose $x = 200$ and $y = 199$. Thus, $(y + x)/(y - x) = 399$.

S94S22 a must obviously be even, so let a=2k. This yields $(2k - 8)(2k - 10) = 2^b$ or $(k - 4)(k - 5) = 2^{b-2}$. Now $(k - 4)$ and $(k - 5)$ are consecutive integers thus one is odd and must be equal to ±1. So $k - 1 = \pm 1$ and $k - 5 = \pm 1$ gives $k = 3, 4, 5,$ or 6. Checking we find that only $k = 3$ and 6 work corresponding to $(a,b) = (6,3)$ or $(12,3)$.

S94S23 There are $x^2/100$ ounces of salt in the solution. Therefore, the new solution yields $(x^2/100)/(x + y) = (x - 30)/100$ or $x^2 = (x + y)(x - 30)$. Solving for y, we obtain $y = 30x/(x - 30)$.

S94S24 We have $x^2 - 14x + 42 = T$ and thus $T^3 = 63T + 162$. $T^3 - 63T - 162 = 0$ or $(T-9)(T^2 + 9T + 18) = 0$ or $(T-9)(T+3)(T+6) = 0$. Hence, $X^2 - 14x + 33 = 0$ and $X^2 - 14x + 48 = 0$ and $X^2 - 14x + 45 = 0$. Solve and the answer is 3, 5, 6, 8, 9, and 11.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER FIVE SPRING, 1994

S94S25 Since $g((x-4)/3) = x$, we are looking for the inverse function. $y = (x-4)/3$ or $x = (y-4)/3$. This yields $y = 3x + 4$. Thus, $g(x) = 3x + 4$.

S94S26 $(x + y + \sqrt{xy})^2 = 45^2 =$

$x^2 + y^2 + xy + 2xy + 2x\sqrt{xy} + 2y\sqrt{xy}$

$1992 + 2\sqrt{xy}\cdot[\sqrt{xy} + x + y] = 1992 + 2\sqrt{xy}\cdot(45)$

$1992 + 2\sqrt{xy}\cdot(45) = 45^2$

$\sqrt{xy} = (45^2 - 1992)/90 = 33/90 = 11/30$. Thus,

$xy = (11/30)^2 = 121/900$

S94S27 The given is equivalent to $\log_b x = a^c$ and this is equivalent to $x = b^{a^c}$.

S94S28 Let one side be x and the remaining side be 27-x. Using Heron's formula, Area $= \sqrt{24\cdot3\cdot(24-x)\cdot(x-3)} = 6\sqrt{2(24-x)(x-3)}$. Now the only permissible value of x which will make the area integral is $x = 10$.

S94S29 $ab/(a+b) = 3$. $ab = 3a + 3b$. $ab - 3a - 3b + 9 = 9$. $(a-3)(b-3) = 9$. Thus, $(a-3,b-3) = (1,9), (3,3), (9,1)$ or $(a,b) = (4,12), (6,6), (12,4)$. Therefore, there are 3 ordered pairs.

S94S30 We have $\cos^2 10^\circ - (\sqrt{3}/2)\cos 10^\circ + \cos^2 20^\circ + 3/16 =$

$\cos 10^\circ(\cos 10^\circ - \sqrt{3}/2) + \cos^2 20^\circ + 3/16$

$\cos 10^\circ(\cos 10^\circ - \cos 30^\circ) + \cos^2 20^\circ + 3/16$

Now $\cos(20^\circ-10^\circ) - \cos(20^\circ+10^\circ) = 2\sin 20^\circ \sin 10^\circ$.

$\cos 10^\circ(2\sin 20^\circ \sin 10^\circ) + \cos^2 20^\circ + 3/16$

$\sin^2 20^\circ + \cos^2 20^\circ + 3/16 = 1 + 3/16 = 19/16$.

May 25, 1994

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1994 NYCIML contests that you requested on the application form.

The following are the corrected or alternative answers for the enclosed contests.. The original answers were inaccurate.

	Question	Correct answer
Senior A	S94S1 was eliminated.	It should have read smallest "positive integer".
Senior B	S94B3	120 or 360
	S94B4	(0, -32)
	S94B9	8
Junior	S94J9 was eliminated.	It was an impossible triangle.
	S94J12	(25, 77)

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML