NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 1994**

PART I: **10 Minutes** **NYCIML Contest One** **Spring 1994**

S94J1. A drawer contains four blue pairs of socks, eight brown pairs of socks, two red pairs of socks and three green pairs of socks. The electricity has gone out one evening and Jules must grope for his socks in the dark. What is the minimum number of socks he must draw in order to get a matching pair?

S94J2. A circular cylinder of radius three is inscribed in a sphere of radius five. Compute the volume of the cylinder in terms of π.

PART II: **10 Minutes** **NYCIML Contest One** **Spring 1994**

S94J3. If one leg of a right triangle has length 16 and the lengths of the hypotenuse and remaining leg are consecutive odd integers, find the length of the hypotenuse.

S94J4. $\sqrt{6 + 4\sqrt{2}}$ simplifies to a number of the form $a + b\sqrt{2}$. Find the ordered pair (a,b).

PART III: **10 Minutes** **NYCIML Contest One** **Spring 1994**

S94J5. Point P is located in square ABCD, such that AP=6, PC=8, and DP=5. Find PB to the nearest tenth.

S94J6. If x and y are real numbers such that $|x|y + xy + y^2 = -40$ and $x + |y| - y = 15$, find the maximum value of x+y

Answers

1. 5	**3.** 65	**5.** 8.7
2. 72π	**4.** (2,1)	**6.** 9

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 1994**

PART I: **10 Minutes** **NYCIML Contest Two** **Spring 1994**

S94J7. A sphere of radius 3 is inscribed in a right circular cylinder so that the top and bottom of the cylinder are tangent to the sphere. Compute the volume of the part of the cylinder NOT containing the sphere.

S94J8. $\sqrt{11 - 6\sqrt{2}}$ reduces to $a + b\sqrt{2}$. Find the ordered pair (a,b).

PART II: **10 Minutes** **NYCIML Contest Two** **Spring 1994**

S94J9. In ΔABC, AB = 9, BC = 12. BM is the median and BD is the angle bisector drawn to AC. If DM=2, find AC.

S94J10. Compute the number of values of "a" such that the integer $a^5 + 3$ is divisible by $a^2 + 1$.

PART III: **10 Minutes** **NYCIML Contest Two** **Spring 1994**

S94J11. In a certain right triangle, the length of the hypotenuse is 2 more than the length of a leg. If the perimeter of the triangle is 1012, find the length of the larger leg.

S94J12. Roslyn selected two cards randomly from a standard deck of cards. She reveals to you that at least one of the cards is red. The probability that both are red can be expressed as the fraction $\frac{a}{b}$ where a and b are relatively prime. Find the ordered pair (a,b).

Answers

7. 18π

8. (3,–1)

9. 28

10. 5

11. 483

12. (325,101)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **SPRING 1994**

PART I: **10 Minutes** **NYCIML Contest Three** **Spring 1994**

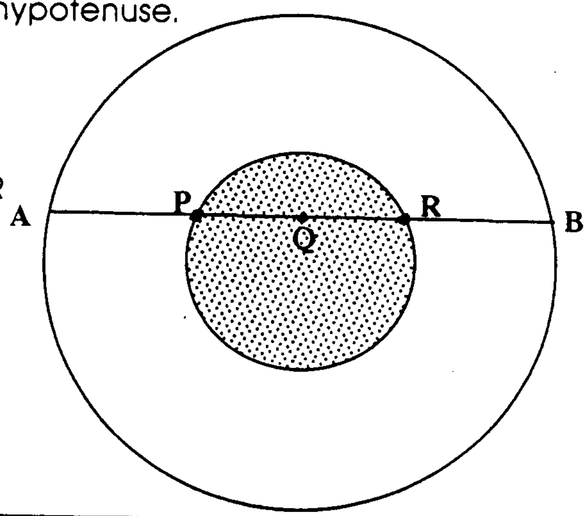
S94J13. $\dfrac{\sqrt{27+10\sqrt{2}}+\sqrt{51-14\sqrt{2}}}{2}$ is an integer x. Compute the value of x.

S94J14. ΔABC is inscribed in a circular dartboard with diameter 10. $m\overset{\frown}{AB}: m\overset{\frown}{BC}: m\overset{\frown}{AC} = 2:3:1$. A dart is thrown randomly at the circle. Find the probability it lands inside the triangular region.

PART II: **10 Minutes** **NYCIML Contest Three** **Spring 1994**

S94J15. In a right triangle, the lengths of a leg and the hypotenuse are odd numbers differing by two. If the area of the triangle is 4080, find the length of the hypotenuse.

S94J16. Find the unshaded area between the concentric circles if AB=12 and P,Q, and R divide $\overline{AB}$ into four congruent segments.



PART III: **10 Minutes** **NYCIML Contest Three** **Spring 1994**

S94J17. A cube is inscribed in a sphere of radius 3. What is the volume of the cube?

S94J18 In triangle ABC, $\overline{BD}$ is the angle bisector, $\overline{BM}$ is the median, and $\overline{BE}$ is the altitude to side AC. If AB = 3, BC = 6, and DM = 1, find ED.

Answers

13. 6

14. $\frac{\sqrt{3}}{2\pi}$ or equivalent

15. 257

16. 27π

17. $24\sqrt{3}$ or equivalent

18. $1\frac{1}{4}$ or equivalent

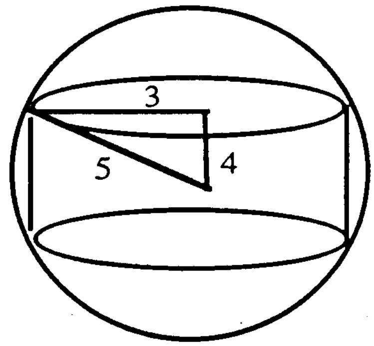
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 1994**

Solutions

S94J1. Since there are 4 different TYPES of socks, choosing 5 socks ensures at least one match. Note that it does not matter how many he has of each type! This method is called the "Pigeon hole" method. **Answer:** 5

S94J2. As the diagram shows, a 3—4—5 right triangle is formed meaning the height of the cylinder is 8. Using the formula $V = \pi r^2 h$, we get $V = \pi(9)8 = 72\pi$. **Answer:** 72π



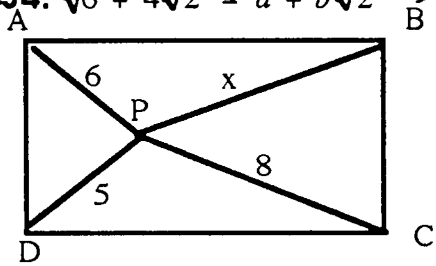
S94J3. Let x and x+2 be the lengths. By the Pythagorean Theorem, we have $16^2 + x^2 = (x+2)^2 \rightarrow 16^2 = 4x + 4$ and $x = 63$ and $x+2 = 65$.

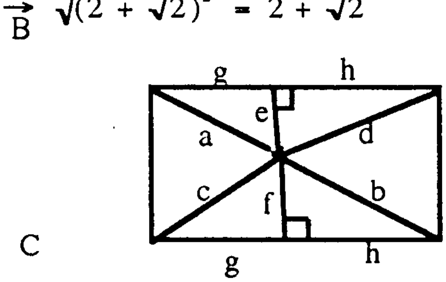
Answer: 65

Note : Generalizing Δs with this property: Let x be an odd whole number. *length of leg #1* is the *sum*: x + (x+2); *length of leg #2* is the *product*: x(x+2) and the *length of the hypotenuse* is x(x+2) +2. [If x is even, the result will be a *non-primitive* triple with the same property.] **Example:** If x=5: leg #1 = 5+7=12, leg #2= 5(7)=35, hypotenuse = 37.

Proof: $(2x+2)^2 + [x(x+2)]^2 = 4x^2+8x+4 + x^4 + 4x^3 + 4x^2 = x^4+4x^3+8x^2+8x+4 = (x^2+2x+2)^2$ ✓

S94J4. $\sqrt{6 + 4\sqrt{2}} = a + b\sqrt{2} \rightarrow \sqrt{(2+\sqrt{2})^2} = 2 + \sqrt{2}$ **Answer:** (2,1)





S94J5. The second diagram generalizes the problem yielding: $a^2 = g^2 + e^2$ and $d^2 = h^2 + e^2$. $\rightarrow a^2 - d^2 = g^2 - h^2$ Likewise: $b^2 = h^2 + f^2$ and $c^2 = g^2 + f^2$. $\rightarrow c^2 - b^2 = g^2 - h^2$. Equating the two expressions gives $a^2 - d^2 = c^2 - b^2$ givingthe interesting result $a^2 + b^2 = c^2 + d^2$. Substituting gives $6^2 + 8^2 = 5^2 + d^2$. so that $5\sqrt{3} \approx 5(1.732) \approx 8.7$ **Answer:** 8.7

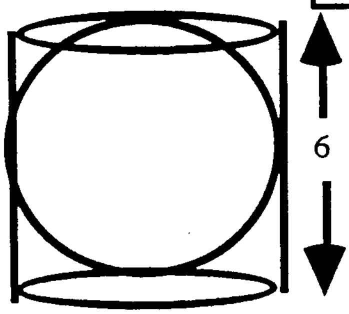
S94J6. In the first equation, $y(|x| + x + y) = -40$. If $x<0$, then $y^2 = -40$ which cannot happen since y is real. Thus $x \geq 0$. In the second equation, if $y > 0$, then $x = 15$. Substituting in the first yields two negative values for y. Thus $y \leq 0$. *Thus x is positive (or 0) and y is negative (or 0).* The two equations become: $2xy + y^2 = -40$ and $x - 2y = 15$. Substitute $x = 15 + 2y$ in the first: $2y(15+2y) + y^2 + 40 = 0$. $5y^2 + 30y + 40 = 0$ so that $(y+4)(y+2) = 0$ and $y = -4$ or $y = -2$. If $y = -4$, $x = 7$ and x+y is 3. If $y = -2$, $x = 11$ and x+y is 9. Thus the maximum value of x+y is 9. **Answer:** 9

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 1994**

Solutions

S94J7.

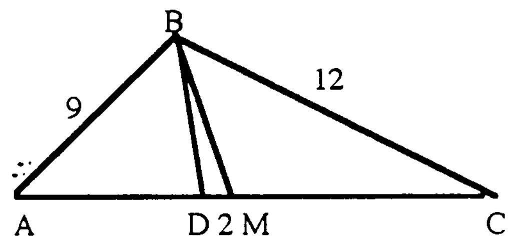


Volume of cylinder=$\pi r^2 h = 54\pi$. Volume of sphere = $\frac{4}{3}\pi r^3 = 36\pi$. The indicated volume is the difference, or 18π.

Answer: 18π

S94J8. $\sqrt{11 - 6\sqrt{2}} = \sqrt{9 - 6\sqrt{2} + 2} = \sqrt{(3 - \sqrt{2})^2} = 3 - \sqrt{2}$ **Answer:** (3,-1)

S94J9. By the angle bisector theorem, let AD=9x and DC=12x.(An angle bisector of a Δ divides the side to which it is drawn in the same ratio as the adjacent sides. Since M is a midpoint, $9x+2 = 12x - 2$ giving $x = \frac{4}{3}$. AC=21x = 28. **Answer:** 28



S94J10. $a^2 + 1 \overline{\smash{)}a^5 + 3}$ with quotient $a^3 - a$

Remainder: a+3

$\frac{a^5+3}{a^2+1} = a^3 - a + \frac{a+3}{a^2+1}$ In order for this result to be an integer, the last fraction must be an integer. This only happens if a = –3, –1, 0, 1, or 2.

Any other value will result in a denominator larger than the numerator. Thus there are 5 values. **Answer:** 5

S94J11. From **S94J3,** the legs can be represented x+x+2, and x^2+2x, and the hypotenuse can be represented by x^2+2x+2. Thus we get $(x^2+2x)+(x^2+2x+2)+(2x+2)=1012$, or $2x^2+6x+4 = 1012$. This equation reduces to $x^2+3x-504=0$ or $(x-21)(x+24)=0$ which means that x = 21. The sides of this triangle have lengths 44,483 and 485. **Answer:** 483

S94J12. When selecting two cards from 52, there are ${}_{52}C_2 = 1326$ outcomes in all. There are ${}_{26}C_2 = 325$ outcomes in which two black were chosen. Since this possibility is out of the question, we deduct 325 from 1326 to get a total of 1001 outcomes in which *at least* one card is red. This gives us the denominator of our answer. There are ${}_{26}C_2 = 325$ outcomes in which two reds were chosen. This gives the numerator. These two numbers are relatively prime. **Answer:** (325,1001)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

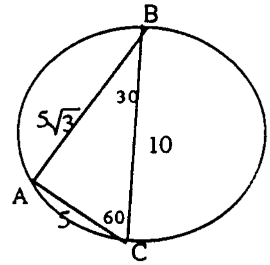
JUNIOR DIVISION | **CONTEST NUMBER THREE** | **SPRING 1994**

Solutions

S94J13.

$$\sqrt{27+10\sqrt{2}} = \sqrt{25+10\sqrt{2}+2} = \sqrt{(5+\sqrt{2})^2} = 5+\sqrt{2}$$
$$\sqrt{51-14\sqrt{2}} = \sqrt{49-14\sqrt{2}+2} = \sqrt{(7-\sqrt{2})^2} = 7-\sqrt{2}$$

The sum of these two expressions is 12. Thus x=6. **Answer:** 6

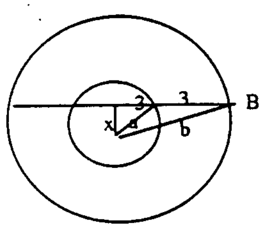


S94J14. Let the measures of the three arcs be 2x,3x and x. 6x=360 so x=60. The three arcs measure 180,120 and 60 meaning the triangle is 30–60–90 with area $\frac{25\sqrt{3}}{2}$. The area of the circle is 25π. Thus the probability of a hit = $\frac{\text{Area}(\Delta)}{\text{Area(circle)}}$

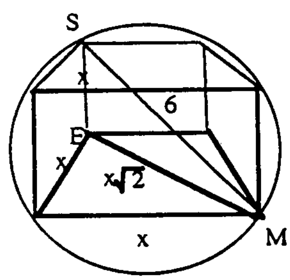
$$= \frac{\frac{25\sqrt{3}}{2}}{25\pi} = \frac{\sqrt{3}}{2\pi}$$

Answer: $\frac{\sqrt{3}}{2\pi}$

S94J15. From **S94J3** and **S94J11**, the lengths of the legs of such a triangle can be written as 2x+2, and x(x+2). The area is therefore $\frac{1}{2}(2x+2)(x)(x+2) = x(x+1)(x+2) = 4080$. The easiest way to solve this is by trial and error! This yields x=15. Thus the length of the hypotenuse is 15(17)+2 = 257.
Answer: 257



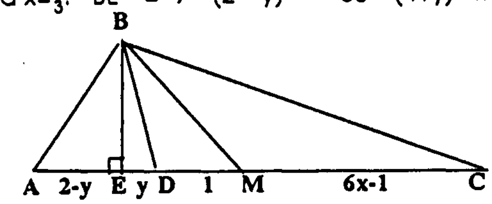
S94J16. Drawing the radii of the two circles and using the Pythagorean Theorem, we get: $a^2 = x^2 + 9$ and $b^2 = x^2 + 36$. Subtracting these equations gives $b^2 - a^2 = 27$. The desired area is $\pi b^2 - \pi a^2 = \pi(b^2 - a^2) = 27\pi$ **Answer:** 27π



S94J17. In ΔSEM, the hypotenuse has length 6. But by the Pythagorean Theorem, this length is $x\sqrt{3}$. The desired volume is x^3. One way to compute this is by taking $x\sqrt{3} = 6$ and cubing both sides to get $3\sqrt{3}x^3 = 216$ so that $x^3 = 24\sqrt{3}$ or equivalent.

Answer: $24\sqrt{3}$

S94J18. By the angle bisector theorem, let AD=3x and DC= 6x so 6x−1 = 3x+1 and $x=\frac{2}{3}$. $BE^2 = 9-(2-y)^2 = 36-(4+y)^2$ which leads to $y = 1\frac{1}{4}$



Answer: $1\frac{1}{4}$

May 25, 1994

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1994 NYCIML contests that you requested on the application form.

The following are the corrected or alternative answers for the enclosed contests.. The original answers were inaccurate.

	Question	Correct answer
Senior A	S94S1 was eliminated.	It should have read smallest "positive integer".
Senior B	S94B3	120 or 360
	S94B4	(0, -32)
	S94B9	8
Junior	S94J9 was eliminated.	It was an impossible triangle.
	S94J12	(25, 77)

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML