NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER ONE FALL, 1993

PART I: TIME: 10 MINUTES

F93B1 Two sides of a <u>scalene</u> triangle with integral sides are 7 and 11. Compute the number of possible lengths for the third side.

F93B2 Find the first term of an infinite geometric progression whose sum is $2\sqrt{2} + 2$ and whose common ratio is $1/\sqrt{2}$.

PART II: TIME: 10 MINUTES

F93B3 If $\log_{10}(9!) = Q$, express $\log_{10}(10!)$ in terms of Q.

F93B4 Find the perimeter of a rhombus whose diagonals are 8 and 10.

PART III: TIME: 10 MINUTES

F93B5 A cube 5 inches on a side is painted, and then cut into 1 inch cubes. How many of these smaller cubes are painted on exactly 2 sides?

F93B6 Triangle ABC is an equilateral triangle with side of length 2. $\overline{BC}$ is extended its own length to point D such that C is the midpoint of $\overline{BD}$. E is the midpoint of $\overline{AB}$, and $\overline{DE}$ intersects $\overline{AC}$ at F. Find the area of quadrilateral BCFE.

<u>ANSWERS</u>

1. 11
2. $\sqrt{2}$
3. $1 + Q$
4. $4\sqrt{41}$
5. 36
6. $(2/3)\sqrt{3}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER TWO FALL, 1993

PART I: TIME: 10 MINUTES

F93B7 Find the result if the sum of the first 100 odd positive integers is subtracted from the sum of the first 100 even positive integers.

F93B8 Find the value of

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

PART II: TIME: 10 MINUTES

F93B9 A man pulls 2 socks at random from a drawer which contains 9 white and 7 black socks. What is the probability that the socks match?

F93B10 Find the product of the 5 terms of a geometric progression whose first term is 1 and whose fifth term is 3.

PART III: TIME: 10 MINUTES

F93B11 Two roots of the equation $x^3 + ax^2 + 17x + b = 0$ are 1 and 2. Find the third root.

F93B12 Find the length of a side of a regular octagon which is inscribed in a circle with radius 1.

ANSWERS

7\. 100

8\. $-1 + \sqrt{2}$

9\. 19/40

10\. $9\sqrt{3}$

11\. 5

12\. $\sqrt{2 - \sqrt{2}}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER THREE FALL, 1993

PART I: TIME: 10 MINUTES

F93B13 $5^4 + 5^4 + 5^4 + 5^4 + 5^4 = 5^x$. Find x.

F93B14 There are 8 good and 4 bad light bulbs in a box. If 3 are picked at random, find the probability that at least 2 are bad.

PART II: TIME: 10 MINUTES

F93B15 If $\log_a 12 = x$ and $\log_a 4 = y$, express $\log_a(a/3)$ in terms of x and y.

F93B16 Find the area enclosed by $|x| + |y| = 1$.

PART III: TIME: 10 MINUTES

F93B17 When $2x^5 + 5x^4 - 11x^3 + ax$ is divided by $x+1$, the remainder is 3. Find a.

F93B18 If $\sin(x/2) = 1/4$, compute $\cos 2x$.

ANSWERS

13. 5
14. 13/55
15. $1 - x + y$
16. 2
17. 11
18. 17/32

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR FALL, 1993

PART I: TIME: 10 MINUTES

F93B19 Compute the sum of all 4 digit numbers which can be formed using the digits 1, 2, 3, and 4, with no repetitions.

F93B20 Find the area of the region bounded by the graphs of $x=5$, $x=0$, $y = -2$ and $y = |2x - 3|$.

PART II: TIME: 10 MINUTES

F93B21 Find all values of x for which $2/(1 - 1/(x - 3))$ has no meaning.

F93B22 A canoeist can row 9 mph in still water. If he takes 3/5 the time to row 40 miles downstream than to row upstream, what is the rate of the current?

PART III: TIME: 10 MINUTES

F93B23 In rectangle ABCD, AB = 15 and BC = 20. F and E are on diagonal BD so that BF = 5 and DE = 5. Find the area of triangle AFE.

F93B24 Find all values of x, $0^\circ \le x \le 360^\circ$, such that $\sin 2x < \sin x$.

ANSWERS

19. 66,660
20. 24.5
21. 3 and 4
22. 2.25
23. 90
24. $60^\circ < x < 180^\circ$ and $300^\circ < x < 360^\circ$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FIVE FALL, 1993

PART I: TIME: 10 MINUTES

F93B25 Find the units digit of the number 3^{1993}.

F93B26 If $\log_5 x = Q$, express $\log_{25} x^2$ in terms of Q.

PART II: TIME: 10 MINUTES

F93B27 How many different positive amounts of money can be made using one or more of the following coins? A penny, a nickel, a dime, a quarter, a half dollar, and a silver dollar.

F93B28 A regular octagon is inscribed in a circle with radius 6. Find the total area outside the octagon and inside the circle.

PART III: TIME: 10 MINUTES

F93B29 Solve for x: $x[x] = 10$.
([x] represents the greatest integer less than or equal to x.)

F93B30 Up until today's game, David had hit exactly 75% of his foul shots. Even though he only hit 3 out of 5 today, his average to the nearest per cent, was still 75%. What is the fewest number of shots he could have taken this season, including today's game, for this to be possible?

ANSWERS

25. 3
26. Q
27. 63
28. $36\pi - 72\sqrt{2}$
29. 10/3
30. 153

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST ONE FALL 1993

F93B1 Since the sum of two sides is greater than the third, and it is scalene, the possible lengths of the third side are 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, and 17. Therefore, there are 11 possible lengths.

F93B2 Using the formula $S = a/(1 - r)$, $2\sqrt{2} + 2 = a/(1 - 1/\sqrt{2}) = a\sqrt{2}/(\sqrt{2} - 1)$. $a\sqrt{2} = 2$ or $a = 2/\sqrt{2} = \sqrt{2}$.

F93B3 $\log_{10}(10!) = \log_{10}(10 \cdot 9!) = \log_{10}10 + \log_{10}(9!) = 1 + Q$.

F93B4 Since the diagonals are perpendicular to each other, and bisect each other, each side is $\sqrt{4^2 + 5^2} = \sqrt{41}$. Therefore, $P = 4\sqrt{41}$.

F93B5 There are 12 edges, each of which has its three interior cubes painted on exactly 2 sides. Since there is no intersection of these cubes, there are 36 cubes.

F93B6 Draw BF. FC = (1/3)AC since F is the point of intersection of the medians of triangle ABD.

Triangle FBC = (1/3) Triangle ABC.

Triangle FEB = (1/2) Triangle AFB = (1/3) Triangle ABC.

Quadrilateral BCFE = (2/3) Triangle ABC = $(2/3)\sqrt{3}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST TWO FALL 1993

F93B7 Separating the numbers into 100 pairs,
$(2-1) + (4-3) + \ldots = 100$.

F93B8 Let $x = \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}$. Then $x = 1/(2 + x)$.

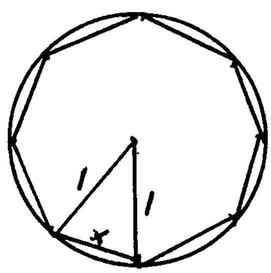
$x^2 + 2x - 1 = 0$ which yields $x = (-2 \pm \sqrt{8})/2 = -1 \pm \sqrt{2}$.
Since x is positive, $x = -1 + \sqrt{2}$.

F93B9 P(2 white) = (9/16)(8/15) = 3/10
P(2 black) = (7/16)(6/15) = 7/40
P(match) = 3/10 + 7/40 = 19/40

F93B10 The common ratio is $3^{1/4}$.
$P = 1\cdot 3^{1/4}\cdot 3^{2/4}\cdot 3^{3/4}\cdot 3 = 3^{5/2} = 9\sqrt{3}$

F93B11 17 = sum of the roots in pairs. Let r be the third root.
$1\cdot 2 + 2\cdot r + 1\cdot r = 17$. $3r = 15$ or $r = 5$.

F93B12 $\sin 22.5^\circ = x/2 = \sqrt{1 - \cos 45^\circ/2} = (\sqrt{2 - \sqrt{2}})/2$.
Therefore, $x = \sqrt{2 - \sqrt{2}}$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST THREE FALL 1993

F93B13 $5^4 + 5^4 + 5^4 + 5^4 + 5^4 = 5 \cdot 5^4 = 5^5$. $x = 5$.

F93B14 The probability is $(8 \cdot ({}_4C_2) + ({}_4C_3))/{}_{12}C_3 =$ $(48 + 4)/220 = 13/55$.

F93B15 $\log_a(a/3) = \log_a a - \log_a 3 = 1 - (\log_a 12 - \log_a 4) =$ $1 - x + y$.

F93B16 The graph is a square with side of length $\sqrt{2}$. $A = (\sqrt{2})^2 =$ 2.

F93B17 Using The Remainder Theorem, the remainder is $f(-1)$.
$2(-1)^5 + 5(-1)^4 - 11(-1)^3 - a = 3$.
$-2 + 5 + 11 - a = 3$, thus $a = 11$.

F93B18 $\sin^2(x/2) = 1/16 = (1 - \cos x)/2$.
$\cos x = 14/16 = 7/8$
$\cos 2x = 2 \cdot \cos^2 x - 1 = 2 \cdot (49/64) - 1 = 34/64 = 17/32$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST FOUR FALL 1993

F93B19 There are 24 numbers, and each digit will appear 6 times in each column. $6(4 + 3 + 2 + 1) = 60$. $60(1111) = 66,660$.

F93B20 The area consists of two right triangles plus one rectangle. $A = (1/2)\cdot 3\cdot (3/2) + (1/2)\cdot (7/2)\cdot 7 + 2\cdot 5$
$= 9/4 + 49/4 + 10 = 24.5$

F93B21 Obviously $x = 3$ will make the fraction meaningless. Also, when simplified, the fraction becomes $2(x - 3)/(x - 4)$, making $x = 4$ also an impossibility.

F93B22 Let x = the rate of the current. Using $D/R = T$,
$40/(9 + x) = (3/5)\cdot(40/(9 - x))$ or $360 - 40x = 216 + 24x$ or
$x = 2.25$

F93B23 Since BD = 25, FE = 15 and the area of triangle AFE is 3/5 the area of triangle ADB which is 1/2 the area of the rectangle $(3/5)(1/2)\cdot 300 = 90$.

F93B24 $2\sin x\cos x < \sin x$ or $2\sin x\cos x - \sin x < 0$
$\sin x(2\cos x - 1) < 0$. This will be true if one is positive and the other is negative. If $\sin x > 0$ and $\cos x < 1/2$,
$60^\circ < x < 180^\circ$. If $\sin x < 0$ and $\cos x > 1/2$,
$300^\circ < x < 360^\circ$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST FIVE FALL 1993

F93B25 The units digit of 3^x runs in cycles of 3, 9, 7, and 1. Since $1993 \equiv 1 \bmod 4$, the units digit is 3.

F93B26 Since $5^Q = x$, $(5^2)^Q = x^2$. $\log_{25} x^2 = Q$.

F93B27 The number of subsets of N objects is 2^N. Since 0 cents is ruled out, and there is no intersection (duplication) of the sets, the number is $2^6 - 1 = 63$.

F93B28 The area of the circle is 36π. The area of the octagon, with 8 congruent triangles, is $8\cdot(1/2)\cdot 6\cdot 6\cdot \sin 45^\circ = 72\sqrt{2}$. Area $= 36\pi - 72\sqrt{2}$.

F93B29 The number is obviously between 3 and 4. Therefore, $[x] = 3$. $3x = 10$ or $x = 10/3$.

F93B30 Let x be the number of shots taken before today. Since David's average still rounds off to 75%, it must be at least .745. Therefore, $(.75x + 3)/(x + 5) \geq .745$ or $x \geq 145$. After today, he has taken 148 + 5 = 153 shots.