NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER ONE

PART I: TIME: 10 MINUTES FALL, 1993

F93S1 A train travels a round trip distance of D miles each way. The rate of the train going is 30 mph and the rate of the train returning is 60 mph. Compute the average rate for the round trip.

F93S2 $\overline{AB}$ and $\overline{CD}$ are chords of a circle which intersect at E. AE, CE, DE, and BE are respectively the fifth, seventh, eleventh, and fourteenth terms of an arithmetic progression. If AB>CD, find AB:CD in terms of positive relatively prime integers. (Find all possible values.)

PART II: TIME: 10 MINUTES FALL, 1993

F93S3 If A is an acute angle and sin2A = x, compute sinA + cosA in terms of x.

F93S4 Suppose A varies directly as the log of B and inversely as the cube of c. Further $A=A_1$ when $B=B_1$ and $C=C_1$.

Also, $A=A_2$ when $B=B_1^{245}$ and $C=105C_1$. Find A_1/A_2.

PART III: TIME: 10 MINUTES FALL, 1993

F93S5 Compute the maximum area of a triangle if the length of one side is 6 and the sum of the lengths of the other two sides is 10.

F93S6 Suppose G is a geometric sequence with ratio r where $-1 < r < 1$. The infinite sum of the members of G is equal to the infinite sum of the cubes of the members of G. If the first term is positive then the value of this term can assume values of T such that $A \leq T < B$. Find B-A.

ANSWERS

1. 40 mph
2. 33:32
3. $\sqrt{1 + x}$
4. 4725
5. 12
6. $\sqrt{3}/2$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER TWO

PART I: TIME: 10 MINUTES FALL, 1993

F93S7 Find the number of terms in the expansion of $[(x + 2y)^2(x - 2y)^2]^3$.

F93S8 A game is played by rolling a die three times <u>or</u> until a six is rolled. Compute, in simplest form, the probability of obtaining at least one five?

PART II: TIME: 10 MINUTES FALL, 1993

F93S9 Compute the maximum value of 17sin3xcos3x.

F93S10 In regular hexagon ABCDEF, triangles ACE and BDF are drawn producing another regular hexagon. Find the ratio of the area of ABCDEF to the area of the new regular hexagon formed.

PART III: TIME: 10 MINUTES FALL, 1993

F93S11 Compute the number of distinct positive integral divisors of $(60)^3$.

F93S12 The roots of $x^3 - 19x^2 - 1992x - 92 = 0$ are A, B, and C. Find [ABC + (A+B)(B+C)(C+A)]/(A+B+C).

ANSWERS

7. 7	9. 17/2	11. 112
8. 19/54	10. 3	12. -1992

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE

PART I: TIME: 10 MINUTES — FALL, 1993

F93S13 Find two numbers a and b such that a-b, ab and a/b are equal. Express the answer as (a,b).

F93S14 Compute the value of $\dfrac{(1^{-2} + 3^{-2} + 5^{-2} + 7^{-2} + \ldots)}{(1^{-2} + 2^{-2} + 3^{-2} + 4^{-2} + \ldots)}$.

PART II: TIME: 10 MINUTES — FALL, 1993

F93S15 An urn contains ten marbles which are numbered one through ten. Terri randomly removes two marbles. Compute the probability that the sum of the two numbers on the marbles is even.

F93S16 Parallelogram ABCD has AB = 2, BC = 7, and the cosine of angle ABC = -1/3. $\overline{ACC'}$ is a line segment with $C'C/\overline{AC} = 11$. $\overline{ABB'}$ is a line segment with $\overline{C'B'}$ perpendicular to B'A. What is the length of BB'?

PART III: TIME: 10 MINUTES — FALL, 1993

F93S17 Three numbers are in increasing arithmetic sequence and three different numbers form a geometric sequence. When the corresponding terms of these two sequences are added successively, the sums are 85, 76, and 84, respectively. If all three terms of the arithmetic sequence are added, the sum is 126. Find the terms of the increasing arithmetic sequence.

F93S18 A rectangle has side lengths of 1 and 2. Four circles of radius 1 are drawn each centered at a different corner of the rectangle. The area of the region which is interior to the rectangle and exterior to each of the circles has area $a + b\sqrt{3} + c\pi$ where a,b, and c are rational. Compute a+b+c.

<u>ANSWERS</u>

13. (-1/2, -1)
14. 3/4
15. 4/9
16. 50
17. 17, 42, 67
18. 7/6

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR

PART I: TIME: 10 MINUTES FALL, 1993

F93S19 Solve $\sin 75^\circ - \cos 75^\circ = \sin x$ for x if $0^\circ \le x \le 360^\circ$.

F93S20 Point E is interior to rectangle ABCD. Find x if $AE = 8x^2-11$, $BE = 4x^2+11$, $CE = 16x$ and $DE = 10x$.

PART II: TIME: 10 MINUTES FALL, 1993

F93S21 Solve in simplest form for a in terms of b and c if $\log_{10}a = c - \log_{10}b$.

F93S22 Al and Bob are respectively 17 and 12 years old and share the same birthday each year. On this birthday, Al and Bob inherit $1,044,204.00 and it is distributed such that each will receive the same amount at age 20 if it is invested now at 10% interest each year. How much is invested for Al now?

PART III: TIME: 10 MINUTES FALL, 1993

F93S23 When Jessica was asked the exact time, she said if you add one-eighth of the time from noon until now to one-quarter the time from now until noon tomorrow you obtain the exact time. Compute the exact time.

F93S24 Triangle ABC is inscribed in circle O of radius one. Angle A measures 30°, angle B measures 70° and angle C measures 80°. M and N are points on circle O such that M is closer to C than to B and $\overline{AM}$ and $\overline{AN}$ trisect angle A. The area of quadrilateral BCMN is $P\sin 20^\circ - \sqrt{Q}$ with P, Q positive rationals. Find (P,Q).

ANSWERS

19. 135°, 45°
20. 3/2
21. $10^c/b$
22. 644,204
23. 5:20 PM
24. (3/2, 3/16)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER FIVE

PART I: TIME: 10 MINUTES FALL, 1993

F93S25 Compute the number of integers which satisfy $x^2 - 24 < 5x$.

F93S26 Let set A be $\{-10, -9, -8, \ldots, -1\}$ and set B be $\{1, 2, 3, \ldots, 10\}$. What is the probability that the roots of a quadratic equation are real given that their sum is in set A and their product is in set B?

PART II: TIME: 10 MINUTES FALL, 1993

F93S27 If $i^2 = -1$, compute the numerical value of $(1 + i)^{40} - (1 - i)^{40}$.

F93S28 One side of a parallelogram is length one unit less than the other side and two and four units less than the diagonals. Find the length of this side.

PART II: TIME: 10 MINUTES FALL, 1993

F93S29 Find the base b such that $73_b = 2(37_b)$.

F93S30 Find all possible triples (x,y,z) such that:

$\log_2 x \cdot \log_2 y + \log_2 xy = 2$

$\log_2 y \cdot \log_2 z + \log_2 yz = 59$

$\log_2 z \cdot \log_2 x + \log_2 zx = 4$

ANSWERS

25. 10
26. 31/50
27. 0
28. $2 + \sqrt{13}$
29. 11
30. $(\sqrt{2}/2, 32, 512)$ and $(\sqrt{2}/4, 1/128, 1/2048)$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER ONE FALL, 1993

F93S1 The average rate can be determined by taking the Harmonic Mean between the two rates. Thus, the average rate is $2(30)(60)/(30 + 60) = 40$ mph.

F93S2 Let AE, CE, DE, and BE equal a+4d, a+6d, a+10d, and a+13d. We know that (AE)(EB) = (CD)(ED) or $(a+4d)(a+13d) = (a+6d)(a+10d)$ or $17ad + 52d^2 = 16ad + 60d^2$ or $d(a-8d) = 0$ which yields d=0 or a=8d. If d=0 then AB:CD = 2a:2a = 1:1. If d does not equal 0 then a=8d which yields AB:CD = [(8d + 4d) + (8d + 13d)]:[(8d + 6d) + (8d + 10d)] = 33d:32d = 33:32. The answer is therefore 33:32.

F93S3 sin2A = 2sinAcosA. Now $(\sin A + \cos A)^2 = 1 + 2\sin A\cos A$. Therefore, $\sin A + \cos A = \sqrt{1 + x}$.

F93S4 We have $A_1C_1{}^3/\log B_1 = A_2(105C_1)^3/\log B_1{}^{245}$.

Thus, $A_1/A_2 = 105^3\log B_1/245\log B_1 =$

$3^3 \cdot 5^3 \cdot 7^3/5 \cdot 7^2 = 4725$.

F93S5 Using Hero's Formula, $k = \sqrt{s(s-a)(s-b)(s-c)}$ where k is the area of the triangle, s is the semiperimeter, and a, b, and c are the lengths of the sides of the triangle, we obtain k $= \sqrt{8(8-6)(8-b)(8-(10-b))} = 4\sqrt{(8-b)(b-2)}$. Since (8-b)+(b-2) is constant, the maximum value occurs when 8-b = b-2 or b=5. Hence, $k = 4\sqrt{3 \cdot 3} = 12$.

F93S6 We have $T/(1 - r) = T^3/(1 - r^3)$ or $T^2 = r^2 + r + 1$. Now $\max(T^2)$ is obviously when r approaches 1, thus $T^2 < 1^2 + 1 + 1 = 3$. Now $T^2 = (r + 1/2)^2 + 3/4$, so $\min(T^2 = 3/4$ when $r = -1/2$. Hence $3/4 \le T^2 < 3$ or $\sqrt{3}/2 \le T < \sqrt{3}$. Thus, the answer is $\sqrt{3} - \sqrt{3}/2 = \sqrt{3}/2$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

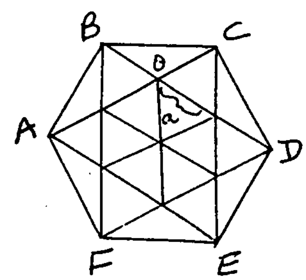
SOLUTIONS CONTEST NUMBER TWO FALL, 1993

F93S7 The given expression can be rewritten as $[(x^2 - 4y^2)^2]^3 = (x^2 - 4y^2)^6$. From the Binomial Expansion Formula, we have 6+1 = 7 terms in the expansion of $(x^2 - 4y^2)^6$.

F93S8 Let's find the probability of the "game" ending without a five occurring. There are three possible cases. Firstly, one could roll a six with a probability of 1/6. Secondly, one could roll a 1, 2, 3, 4; roll a 6 with a probability of (4/6)(1/6) = 1/9. Thirdly, one could roll a 1, 2, 3, 4; roll 1, 2, 3, 4; roll 1, 2, 3, 4, 6 with a probability of (4/6)(4/6)(5/6) = 10/27. Thus the answer is 1 - 1/6 - 1/9 - 10/27 = 19/54.

F93S9 Using the identity sin2A = 2sinAcosA, we obtain 17sin3xcos3x =(17/2)·sin6x. Hence, the maximum value of sin6x is 1 and therefore, the maximum value of 17sin3xcos3x is 17/2.

F93S10 Draw the three shown diagonals of the smaller hexagon and let its side be a. Also notice that θ is 120°. Now the entire hexagon is made up of 12 equilateral triangles of side length a and six isosceles triangles with legs a and vertex angle of θ. The new hexagon is formed of six equilateral triangles of side length a.



Thus the answer is

$[12\cdot(1/2)\cdot a\cdot a\cdot \sin 60^\circ + 6\cdot(1/2)\cdot a\cdot a\cdot \sin 120^\circ]/$
$[6\cdot(1/2)\cdot a\cdot a\cdot \sin 60^\circ] = [9a^2 \sin 60^\circ]/3a^2 \sin 60^\circ] = 3.$

F93S11 $60 = 2^2\cdot 3\cdot 5$ and $(60)^3 = 2^6\cdot 3^3\cdot 5^3$. Therefore, the number of divisors is (6+1)(3+1)(3+1) = 7·4·4 = 112.

F93S12 First $(A+B)(B+C)(C+A) = A^2B + A^2C + B^2A + B^2C + C^2A + C^2B + 2ABC$. Now, $(A+B+C)(AB+BC+CA) = A^2B + A^2C + B^2A + B^2C + C^2A + C^2B + 3ABC = (A+B)(B+C)(C+A) + ABC$. Thus, we seek AB + BC + CA which is -1992.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER THREE FALL, 1993

F93S13 Let $ab = a/b$ or $ab^2 = a$ or $b = \pm 1$. If $a = 0$, then $b = 0$ and a/b is undefined. Consider $a-b = ab$. If $b=1$, then this implies $a - 1 = a$ which is a contradiction. If $b= -1$, then $a + 1 = -a$ which yields $a = -1/2$. Therefore, $a = -1/2$ and $b = -1$.

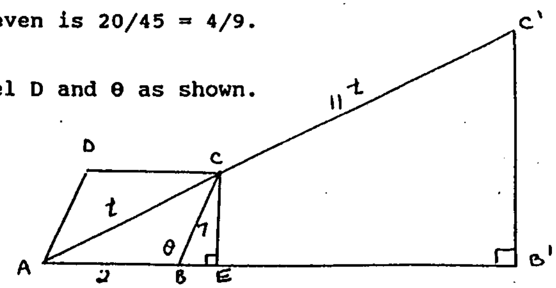
F93S14 The given expression is equivalent to

$$\frac{[1^{-2}+2^{-2}+3^{-2}+4^{2}+\ldots] - [2^{-2}+4^{-2}+6^{-2}+8^{-2}+\ldots]}{1^{-2}+2^{-2}+3^{-2}+4^{2}+\ldots}$$

$= 1 - 2^{-2}[1^{-2}+2^{-2}+3^{-2}+\ldots]/[1^{-2}+2^{-2}+3^{-2}+\ldots]$
$= 1 - 2^{-2} = 3/4$.

F93S15 In order for Terri to obtain an even sum, she must select two marbles that are either both even or both odd. There are a total of ${}_5C_2 + {}_5C_2 = 20$ ways to obtain this outcome. There are a total of ${}_{10}C_2 = 45$ ways to select two marbles from the urn containing 10 marbles. Thus, the probability that the sum of the two numbers on the marbles is even is $20/45 = 4/9$.

F93S16 Label D and θ as shown.



Now $\cos CBE = -\cos\theta = 1/3$. $BE = 7/3$. By similarity $t/12t = (2 + 7/3)/AB'$ or $AB' = 12(2 + 7/3) = 52$.
Now $BB' = 52 - 2 = 50$.

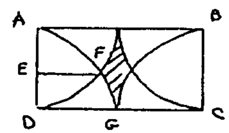
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER THREE FALL, 1993

F93S17 Let $a-d$, a, $a+d$ represent the arithmetic sequence and b/r, b, br represent the geometric sequence. We have,

$a - d + b/r = 85$ (1)
$a + b = 76$ (2)
$a + d + br = 84$ (3)
$3a = 126$ (4)

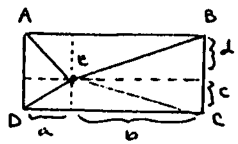
From equation (4), $a = 42$ and from (2) $b = 34$. Adding (1) and (3), we have $2a + b(r + 1/r) = 169$ or $84 + 34(r + 1/r) = 169$ which yields $2r^2 - 5r + 2 = 0$. Solving for r we get $r = 1/2$ or $r = 2$. If $r = 1/2$ then $d = 25$ and if $r = 2$ then $d = -26$. Thus, the required increasing arithmetic sequence is 17, 42, 67.

F9318 AD=1, AB=2, ED=1/2. $EF = \sqrt{1^2 - (1/2)^2} = (1/2)\sqrt{3}$
Shaded Area = A(ABCD) - 2A(Triangle AFD) - 4A(Sector DFG)
A(Triangle AFD) $= (1/2)(1)(\sqrt{3}/2) = \sqrt{3}/4$
Angle FDE$=60^\circ$ and Angle FDG $= 30^\circ$
A(Sector DFG) $= (30/360)(\pi\cdot 1^2) = \pi/12$
Thus, the shaded area $= 2 - 2(\sqrt{3}/4) - 4(\pi/12)$
$= 2 - \sqrt{3}/2 - \pi/3$ and $a+b+c = 2 - (1/2) - (1/3) = 7/6$.



F93S19 $\sin 75^\circ = \sin(30^\circ + 45^\circ)$
$= \sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ$
$\cos 75^\circ = \cos(30^\circ + 45^\circ)$
$= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$
Now, $\sin 75^\circ - \cos 75^\circ = \sin 45^\circ(\sin 30^\circ + \sin 30^\circ)$
$= \sin 45^\circ(2\sin 30^\circ)$ since $\sin 45^\circ = \cos 45^\circ$. Now, $\sin 45^\circ(2\sin 30^\circ) = \sin 45^\circ = \sqrt{2}/2$. Thus, $x = 45^\circ$ or 135°.

F93S20 From E drop perpendiculars to the sides and label lengths as shown. Now



$AE^2 = a^2 + d^2$
$BE^2 = b^2 + d^2$
$CE^2 = b^2 + c^2$
$DE^2 = a^2 + c^2$
We see that $AE^2 + CE^2 = BE^2 + DE^2$. Thus,
$(8x^2-11)^2 + (16x)^2 = (4x^2+11)^2 + (10x)^2$
$64x^4 + 80x^2 + 121 = 16x^4 + 188x^2 + 121$
$48x^4 = 108x^2$
$12x^2(4x^2-9) = 0$ or $x = 3/2$.

F93S21 $c = \log_{10}a + \log_{10}b$ or $c = \log_{10}ab$.

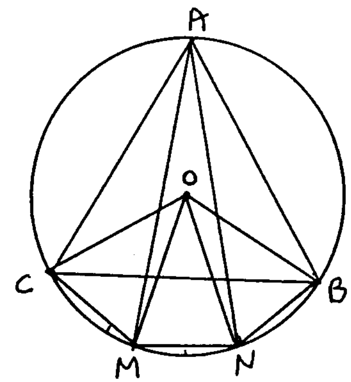
$ab = 10^c$ and therefore, $a = 10^c/b$.

F93S22 Let the amounts be A and B. Then $A+B = 1044204$ and $(11/10)^3A = (11/10)^8B$ or $(10^5/11^5)A = B$.
Now $A(1 + 10^5/11^5) = 1044204$ or
$A = (11^5 \cdot 1044204)/(11^5 + 10^5)$ and $A = 11^5 \cdot 4$ or $A = \$644204$.

F93S23 Let x represent the time interval from noon to now in hours. Hence, $(1/8)x + (1/4)(24 - x) = x$. Solving for x we obtain, $x = 16/3 = $ 5:20 PM.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER FOUR FALL, 1993

F93S24 Let O be the center then angle COM = angle NOB $= 20^\circ$, angle COB $= 60^\circ$ and CO = MO = NO = BO = 1. Thus the answer is
$3[(1/2)(1)(1)\sin 20^\circ] - (1/2)(1)(1)\sin 60^\circ$
$= (3/2)\sin 20^\circ - \sqrt{3}/4$
$= (3/2)\sin 20^\circ - \sqrt{3/16}$. Thus the answer is $(3/2, 3/16)$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER FIVE FALL, 1993

F93S25 Consider $x^2 - 5x - 24 = 0$. $(x+3)(x-8) = 0$ which yields solutions of $x= -3$ and $x=8$. Therefore, all integers N such that $-3 < N < 8$ satisfy $x^2 - 24 < 5x$. Thus, there are 10 solutions.

F93S26 Let $a \in A$ be the sum and let $b \in B$ be the product. Now the quadratic equation is $x^2 - ax + b = 0$ and the roots are real iff $a^2 - 4b \geq 0$ or $b \leq a^2/4$.

$a \in A$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
# $b \in B$ s.t. $b \leq a^2/4$	10	10	10	10	9	6	4	2	1	0

Thus 62 pairs from the 100 pairs in AxB give real roots. The answer then is 62/100 = 31/50.

F93S27 $((1 + i)^2)^{20} - ((1 - i)^2)^{20} = (2i)^{20} - (-2i)^{20} =$ 0.

F93S28 By a common theorem,

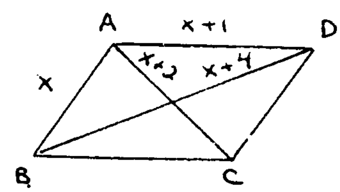
$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

$2(x^2 + (x+1)^2) = (x+2)^2 + (x+4)^2$

$4x^2 + 4x + 2 = 2x^2 + 12x + 20$

$x^2 - 4x - 9 = 0$ and therefore, $x = 2 \pm \sqrt{13}$.

We seek the solution $2 + \sqrt{13}$.



F93S29 $7b + 3 = 2(3b + 7)$ or $b = 11$.

F93S30 Let $\log_2 x$, $\log_2 y$, and $\log_2 z$ equal a,b, and c respectively. Then, $ab+a+b+1 = 2+1$ or $(a+1)(b+1) = 3$. Likewise, $(b+1)(c+1) = 60$ and $(c+1)(a+1) = 5$. Multiply to obtain $(a+1)^2(b+1)^2(c+1)^2 = 900$ or $(a+1)(b+1)(c+1) = \pm 30$. $a+1 = \pm 30/60$, $b+1 = \pm 30/5$, $c+1 = \pm 30/3$. $(a,b,c) = (-1/2,5,9)$ or $(-3/2,-7,-11)$. Therefore, $(x,y,z) = (\sqrt{2}/2,32,512)$ and $(\sqrt{2}/4,1/128,1/2048)$.