NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 1993**

PART I: 10 Minutes **NYCIML Contest One** **Fall 1993**

F93J1. Lucy baked bread one evening. She used yeast that doubles its weight every five minutes. If her 64 ounce pan was filled at 9:03 PM, at what time did the bread reach 4 ounces?

F93J2. Find all real numbers x such that $\sqrt{9x^2 - 30x + 25} = 3x - 5$

PART II: 10 Minutes **NYCIML Contest One** **Fall 1993**

F93J3. In a right triangle, the longest side has length $p^2 + q^2$ while the shortest side has length $p^2 - q^2$. Find in terms of p and q the length of the remaining side.

F93J4. The probability that Deanna gets this problem right is $\frac{5}{8}$. The probability that Judy gets it right is $\frac{3}{8}$. If the probability that Judy gets it right and Deanna gets it wrong is $\frac{1}{4}$, find the probability that both get it wrong.

PART III: 10 Minutes **NYCIML Contest One** **Fall 1993**

F93J5. Steve tested a number N to see if it is prime by dividing the primes 2,3,5,7,... into N. When he reached 103, he finally found that N was NOT prime. Compute the smallest possible value of N.

F93J6. The center of a circle is the origin of a coordinate system. Rectangle OPQR is constructed using point P on the x-axis, 8 units to the right of O(0,0), point R, on the y-axis, and point Q on the circle. If S is the intersection of the circle and the positive x-axis, and PS = 2, find OR.

Answers

1. 8:43 (PM)
2. $\{x \mid x \geq \frac{5}{3}\}$ or equivalent
3. 2pq
4. $\frac{1}{8}$
5. 10609
6. 6

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 1993**

PART I: 10 Minutes **NYCIML Contest Two** **Fall 1993**

F93J7. Two trains heading toward each other on the same straight track are 225 miles apart at 11:45 AM. If these trains travel at *constant* speeds of 80 and 70 miles per hour, at what time will they crash?

F93J8. The probability that Al has seen Jurassic Park is x and the probability that Gerry has seen it is y. If the probability that both have seen the movie is z, find in terms of x, y and z the probability that neither have seen the movie.

PART II: 10 Minutes **NYCIML Contest Two** **Fall 1993**

F93J9. If $a^2b + ab^2 + a + b = 150$ and $2ab = 10$, compute the value of $a^2 + b^2$.

F93J10 A right triangle with hypotenuse of length 149 and whose legs have lengths a and b, where a and b are relatively prime integers and $a<b$. Find the ordered pair (a,b).

PART III: 10 Minutes **NYCIML Contest Two** **Fall 1993**

F93J11. When a certain number is added to both numerator and denominator of $\frac{2}{7}$, the result is the same fraction inverted. Compute the number in question.

F93J12. The sum $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{99\cdot 101}$ can be written in simplest form as $\frac{a}{b}$. Compute the ordered pair (a,b).

Answers

7. 1:15 PM
8. $1 - x - y + z$
9. 615
10. (51,140)
11. -9
12. (50,101)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1993**

PART I: **10 Minutes** **NYCIML Contest Three** **Fall 1993**

F93J13. A one mile long train is about to enter a two mile long tunnel. If the train is traveling at a constant speed of 45 miles per hour and enters the tunnel at 11:55 AM, at what time will the rear of the train leave the tunnel?

F93J14. In an isosceles trapezoid, the length of the longer base is equal to the length of a diagonal. The length of the shorter base is equal to the length of a leg. Find the measure of the smallest angle of the trapezoid.

PART II: **10 Minutes** **NYCIML Contest Three** **Fall 1993**

F93J15. In a right triangle, the length of one leg is $p^2 - 25$. If the three lengths of the triangle are integers with NO common factor, compute the smallest possible length of the hypotenuse.

F93J16. A certain number when added to the fraction $\frac{a}{b}$ where $a \neq b$ inverts the original fraction. The number in question can be written as $m(a+b)$. Compute the value of m.

PART III: **10 Minutes** **NYCIML Contest Three** **Fall 1993**

F93J17. The sum $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{10\cdot 11}$ can be written in simplifiest form as the fraction $\frac{a}{b}$. Find the ordered pair (a,b).

F93J18. If $xy = 5$ and $x+y = 8$, compute the value of $x^5y^2 + 3x^4y^3 + x^4y^2 + 3x^3y^4 + 2x^3y^3 + x^2y^5 + x^2y^4$.

Answers

13. 11:59 (AM)

14. 72

15. 61

16. –1

17. (10,11)

18. 14400

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 1993**

Solutions

F93J1. Working backwards:

At 9:03 → 64 ounces, At 8:58 → 32 ounces, At 8:53 → 16 ounces,
At 8:48 → 8 ounces, At 8:43 → 4 ounces **Answer:** 8:43 (PM)

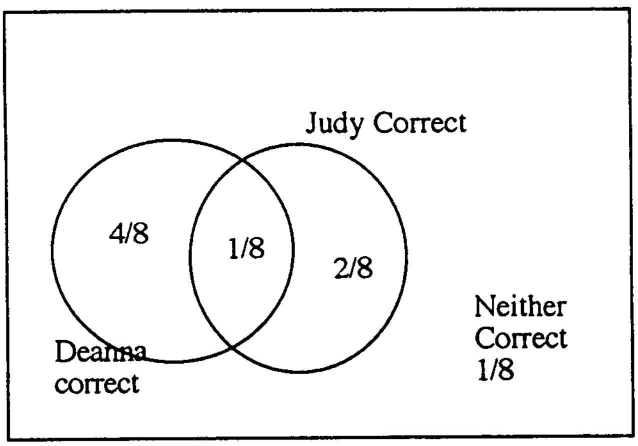
F93J2. $\sqrt{9x^2 - 30x + 25} = 3x - 5$, $\sqrt{(3x-5)^2} = 3x - 5$, $|3x-5| = 3x - 5$

This is true only if $3x - 5 \geq 0$ meaning $x \geq \frac{5}{3}$ **Answer:** $\{x \mid x \geq \frac{5}{3}\}$ or equivalent

F93J3. Let x = the desired length Use the Pythagorean Theorem to get:
$(p^2 + q^2)^2 = (p^2 - q^2)^2 + x^2$, $2p^2q^2 = x^2 - 2p^2q^2 \rightarrow x^2 = 4p^2q^2$
$x = 2pq$ **Answer:** $x = 2pq$

Note: If the lengths of the three sides of a right triangle have NO common factor, then they are in the form $p^2 - q^2$, $2pq$, and $p^2 + q^2$.

F93J4. The Venn diagram below indicates tthe desired probability is $\frac{1}{8}$.



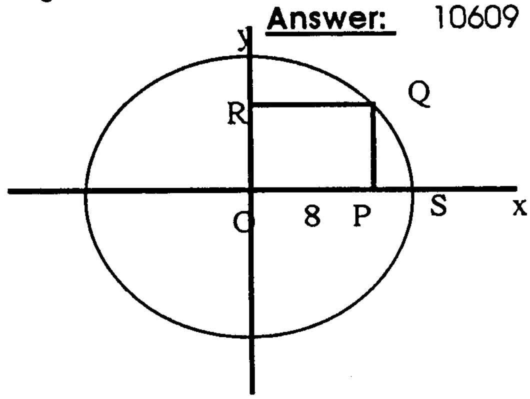
Answer: $\frac{1}{8}$

F93J5. N has 103 as a factor and nothing smaller! The smallest such N is 103^2 or 10609.
Answer: 10609

F93J6. Since the diagonals of a rectangle are ≅, RP= OQ = OS = 10.

R, O, and P form a 6,8,10 triangle so that OR = 6.

Answer: 6



Please note: Concepts used today will be repeated later this year.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 1993**

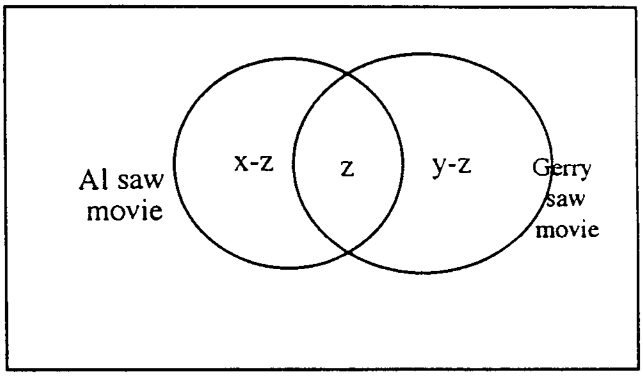
Solutions

F93J7. Let t be the time it takes the two trains. Since they move at constant speeds, the distances traveled can be expressed as 70t and 80t. The sum of these distances is 225 miles. Thus we get the equation 150t=225 and t=1.5. An hour and a half later would be 1:15PM. **Answer:** 1:15 PM.

F93J8. The Venn diagram shows that P(Al or Gerry do see the movie) = $x+y-z$.Thus the probability that both did not see the movie is $1-(x+y-z)$

$= 1-x-y+z$

Answer: $1-x-y+z$



F93J9. Note that $(a+b)^2 = a^2+2ab+b^2 = a^2+b^2+10$. From the given, $a^2b+ab^2+a+b=150 \rightarrow ab(a+b)+(a+b)=150 \rightarrow (ab+1)(a+b)=150$
$6(a+b)=150$ so that $a+b=25$ and $(a+b)^2 = a^2+2ab+b^2 = 625$. Substitute $2ab=10$, $a^2+b^2=615$. **Answer:** 615

F93J10. Using the results from **F93J4**, the hypotenuse has length p^2+q^2 and the two legs have lengths $2pq$ and p^2-q^2. Thus we have $p^2+q^2=149$. The only integers for which this holds is $p=10$ and $q=7$.[Note: Every prime of the form $4n+1$ can be written as the sum of two squares in exactly one way.] Thus $2pq=140$ and $p^2-q^2=51$. **Answer:** (51,140)

F93J11. Let x = the number to be added. This means that $\frac{2+x}{7+x}=\frac{7}{2}$.

$4+2x=49+7x$ so that $x=-9$. (This checks!) **Answer:** -9

F93J12. Note that

$$\frac{1}{1}-\frac{1}{3}=\frac{2}{1\cdot 3}$$
$$\frac{1}{3}-\frac{1}{5}=\frac{2}{3\cdot 5}$$
$$\frac{1}{5}-\frac{1}{7}=\frac{2}{5\cdot 7}$$

and so on • • •

$$\frac{1}{99}-\frac{1}{101}=\frac{2}{99\cdot 101}$$

Adding gives a "telescoping sum" in which many terms drop out.

This gives $\frac{1}{1}-\frac{1}{101}=2\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{99\cdot 101}\right)$

So the desired sum is $\left(\frac{1}{2}\right)\cdot\left(\frac{100}{101}\right)=\frac{50}{101}$ **Answer:** (50,101)

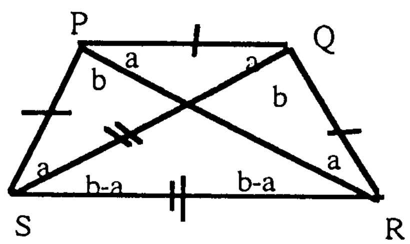
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1993**

Solutions

F93J13. The rear of the train must travel three miles. Since it travels 45 miles in 60 minutes, it will do 3 miles in 4 minutes. Thus the rear of the train emerges at 11:59 A.M. **Answer:** 11:59 (AM)

F93J14.



Using the two isosceles triangles that result, we get: In ΔPQR: $b + 3a = 180$

In ΔPRS: $3b - a = 180$

Solving the two equations in two variables, we get $a = 36$ and $b = 72$. The angles of the trapezoid measure $a+b=108$ and $b=72$. The smallest angle of the trapezoid measures 72. **Answer:** 72

F93J15. The lengths of the three sides can be represented by $p^2 - 25$, $10p$, and $p^2 + 25$. (See questions **F93J4** and **F93J10.**) Since $p > 5$, the smallest value p can have is 6 giving an 11-60-61 right triangle. **Answer:** 61

F93J16. This question generalizes question **F93J11**. Let x = the number to be added. We get $\frac{a+x}{b+x} = \frac{b}{a}$, or $b^2 + bx = a^2 + ax \rightarrow bx - ax = a^2 - b^2$. $x(b-a) = a^2 - b^2 \rightarrow$ Since $a \neq b$, we can divide both sides by $b - a$ giving $x = -(a+b)$ so that $m = -1$ **Answer:** -1

F93J17. Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Thus $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{10\cdot 11} =$
$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{10} - \frac{1}{11}) = 1 - \frac{1}{11} = \frac{10}{11}$ **Answer:** (10,11)

F93J18. Rearranging gives: $x^5y^2+3x^4y^3+x^4y^2+3x^3y^4+2x^3y^3+x^2y^5+x^2y^4$

$= (x^5y^2+3x^4y^3+3x^3y^4+x^2y^5) + (x^4y^2 + x^2y^4 + 2x^3y^3)$

$= x^2y^2 [x^3 + 3x^2y + 3xy^2 + y^3] + x^2y^2 [x^2 + 2xy + y^2]$

$= 25(x+y)^3 + 25(x+y)^2 = 25(x+y)^2(x+y+1) = 25(64)(9) = 14400$ **Answer:** 14400

January 27, 1994

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1993 NYCIML contests that you requested on the application form.

The following are the corrected or alternative answers for the enclosed contests:

	Question	Correct answer
JUNIOR	F93J15	5
	F93J16	(b - a)/ab

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML