NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE

PART I: TIME: 10 MINUTES SPRING, 1993

S93B1 If f(x) = (x+1)/(x-1), find f((x+1)/(x-1)).

S93B2 Find the area bounded by $y = |2x + 3| - 5$ and the x axis.

PART II: TIME: 10 MINUTES SPRING, 1993

S93B3 A circle is inscribed in a regular hexagon. If the area of the circle is 27π, find the area of the hexagon.

S93B4 If $\log_3 625 = a$ and $\log_9 5 = b$, express a in terms of b.

PART III: TIME: 10 MINUTES SPRING, 1993

S93B5 How many ounces of pure acid should be added to 20 ounces of a solution which is 20% acid to make it a 30% solution?

S93B6 Find the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$ in simplest form.

SOLUTIONS

1. x
2. $12\frac{1}{2}$
3. $54\sqrt{3}$
4. 8b
5. 2 6/7
6. 2

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER TWO

PART I: TIME: 10 MINUTES SPRING, 1993

S93B7 Find the remainder when $x^{10} + 2$ is divided by $x-2$.

S93B8 A man put D dollars into a bank. Part of the money went into an account yielding p% interest, and the rest went into an account yielding q% interest. If his total annual income was I dollars, express in terms of D, I, p and q, the amount invested at p%.

PART II: TIME: 10 MINUTES SPRING, 1993

S93B9 How many points are common to the graphs $x^2 = 16$ and $y^2 = 16$.

S93B10 The vertices of triangle ABC are A(0,5), B(3,-1) and C(1,-2). Find the length of the altitude from B to AC.

PART III: TIME: 10 MINUTES SPRING, 1993

S93B11 Find the measure of the angle formed by the hour and minute hands of a clock at 1:05.

S93B12 A girl rolls a fair die, then flips a fair coin, alternately until she either rolls a 4 or flips a tail, at which time her experiment ends. She "wins" if she rolls a 4, and "loses" if she flips a tail. Find the probability that she wins.

SOLUTIONS

7. 1026	9. 4	11. $2\frac{1}{2}^{\circ}$
8. (100I - Dq)/(p - q)	10. $3\sqrt{50}/10$	12. 2/7

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER THREE

PART I: TIME: 10 MINUTES SPRING, 1993

S93B13 Find the sum of the digits of the first 200 positive integers.

S93B14 Solve for all values of x: $2 \cdot 4^{2x} - 17 \cdot 4^{x} + 8 = 0$.

PART II: TIME: 10 MINUTES SPRING, 1993

S93B15 In a fruit store, John buys 5 apples, 8 pears, and 11 oranges and pays \$2.63. Charles buys 3 apples, 5 pears, and 7 oranges and pays \$1.65. If Cathy buys 1 apple, 1 pear, and 1 orange, how much will she pay?

S93B16 If x is acute and $\sin 2x = \frac{1}{4}$, find the value of $\sin x + \cos x$.

PART III: TIME: 10 MINUTES SPRING, 1993

S93B17 Find the number of ordered pairs (x,y), x and y are integers, which satisfy $y = 2x + 1$ and $x^2 + y^2 < 25$.

S93B18 If 796, 1157, and 1594 are divided by the positive integer q, they all leave a remainder of $r > 0$. Find r.

SOLUTIONS

13. 1902
14. $-1/2$, $3/2$
15. 31 cents
16. $\sqrt{5}/2$
17. 4
18. 17

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FOUR

PART I: 10 Minutes SPRING, 1993

S93B19 Find the ordered 4-tuple (a,b,c,d) which satisfies

$$\begin{aligned} 2a + b + c + d &= 32 \\ a + 2b + c + d &= 35 \\ a + b + 2c + d &= 39 \\ a + b + c + 2d &= 44 \end{aligned}$$

S93B20 Find the remainder when 5^{1993} is divided by 7.

PART II: 10 Minutes SPRING, 1993

S93B21 Before the last game of the basketball season, David had hit $37\frac{1}{2}$ % of his shots. On the last day, he hit 11 of his 12 shots and brought his season's percentage up to 40 %. How many shots did he take during the entire season?

S93B22 Solve for all values of x: $\sqrt[3]{x} = 5/(4 + \sqrt[3]{x})$

PART III: 10 Minutes SPRING, 1993

S93B23 Mary added the page numbers of her math book and got a sum of 9000. She added one page twice, however. Which page did she count twice?

S93B24 In triangle ABC, BC=8 and AC=10. If the medians from A and B are perpendicular to each other, find the length of AB.

ANSWERS

19. (2,5,9, 14) 21. 260 23. 89

20. 5 22. −125, 1 24. $\sqrt{164/5}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FIVE

PART I: TIME: 10 MINUTES SPRING, 1993

S93B25 If $i = \sqrt{-1}$, compute $(1 + i)^{12}$.

S93B26 There are 25 marbles in a bag; some are blue and the rest are green. Three marbles at random are removed from the bag (without replacement) and they are all blue. If the probability of drawing 3 blue marbles was 143/1150, how many blue marbles were originally in the bag?

PART II: TIME: 10 MINUTES SPRING, 1993

S93B27 Three congruent circles are placed in an equilateral triangle so that each is tangent to the other two, and each is tangent to two sides of the triangle. If the radius of each circle is 4, find the perimeter of the triangle.

S93B28 If $\log_{10}2=p$ and $\log_{10}3=q$, express $\log_5 72$ in terms of p and q.

PART III: TIME: 10 MINUTES SPRING, 1993

S93B29 If a, b, c, d, and e are all non-zero real numbers, find all possible values of

$a/|a| + b/|b| + c/|c| + d/|d| + e/|e| + abcde/|abcde|$.

S93B30 In triangle ABC, AB=6, BC=8 and the median from B to AC=4. Find the length of AC.

SOLUTIONS

25. -64

26. 13

27. $24 + 24\sqrt{3}$

28. (2q+3p)/(1-p)

29. 6, 2, -2, -6

30. $2\sqrt{34}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

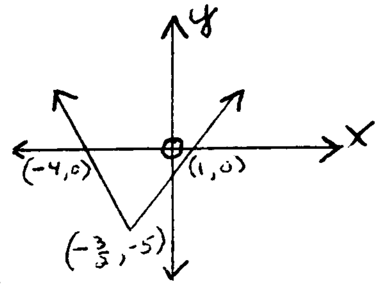
SOLUTIONS CONTEST NUMBER ONE SPRING, 1993

S93B1 Substituting, we have a complex fraction

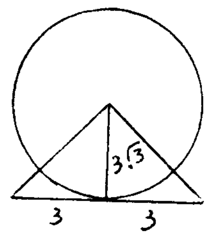
$$\frac{\left(\frac{x+1}{x-1}+1\right)(x-1)}{\left(\frac{x+1}{x-1}-1\right)(x-1)} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

S93B2 This is a triangle with base 5 and height 5.

$A = \frac{1}{2}\cdot 5\cdot 5 = 12\frac{1}{2}$.



S93B3 Since the central angle of the hexagon is 60°, we have six congruent equilateral triangles. Each triangle has height $3\sqrt{3}$ and therefore side 6 and area $9\sqrt{3}$. $6(9\sqrt{3}) = 54\sqrt{3}$.



S93B4 Changing to exponential form, $3^a = 5^4$ and $3^{2b} = 5$. $(3^{2b})^4 = 5^4$, $3^{8b} = 5^4$, $a = 8b$.

S93B5 Equating the number of ounces of acid,

$4 + x = .3(20 + x)$

$40 + 10x = 60 + 3x$

$x = 2\ 6/7$

S93B6 Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$

$x = \sqrt{2 + x}$

$x^2 = 2 + x$

$x^2 - x - 2 = 0$

Rejecting the negative root, $x = 2$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER TWO SPRING, 1993

S93B7 Using the remainder theorem, the remainder is $f(2) = 2^{10} + 2 = 1026$. The same result can be found using synthetic division.

S93B8 Letting x = the amount invested at p%.

$I = px/100 + (D - x)q/100$

$100I = px + Dq - qx$

$x = (100I - Dq)/(p - q)$

S93B9 These are the graphs of the lines $x = \pm 4$ and $y = \pm 4$, intersecting in four points: (4,4), (4,-4), (-4,4), (-4,-4).

S93B10 Since the slopes of AB and BC are negative reciprocals, this is a right triangle with area $\frac{1}{2}\cdot\sqrt{5}\cdot 3\sqrt{5} = 15/2$. Since $AC = \sqrt{50}$, the area also equals $\frac{1}{2}\cdot x\cdot\sqrt{50} = 15/2$. $x = 15/\sqrt{50}$ or $3\sqrt{50}/10$.

S93B11 At 5 minutes after the hour, the clock has moved 1/12 of the way from 1 to 2, or $(1/12)(30^\circ) = 2\frac{1}{2}^\circ$.

S93B12 The probability of winning on the first roll of the die is 1/6. She also wins if she rolls another number, then flips a head, then rolls a 4, or (5/6)(1/2)(1/6). This continues and therefore, forms an infinite geometric progression with common ratio 5/12. Therefore, the sum is (1/6)/(1 - 5/12) = 2/(12 - 5) = 2/7.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER THREE SPRING, 1993

S93B13 The sum of 1 through 9 is 45. In the first hundred integers, there are 10 of these sets in the units digits and 10 of these sets in the tens digits, to total 900. The second hundred integers have the same 900, plus 100 ones in the hundreds place. Add 2 for 200 to get 1902.

S93B14 Let $4^x = y$. Then $2y^2 - 17y + 8 = 0$ or $(2y-1)(y-8) = 0$. Thus $4^x = 1/2$ and $4^x = 8$. $x = -1/2, 3/2$.

S93B15 Let x represent the cost of an apple, y represent the cost of a pear, and z represent the cost of an orange. Thus, $5x + 8y + 11z = 263$ and $3x + 5y + 7z = 165$. Multiplying the first equation by 2 and the second equation by 3 and subtracting yields $x + y + z = 31$.

S93B16 Let $y = \sin x + \cos x$. $y^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x$. $y^2 = 1 + 1/4 = 5/4$. Therefore, $y = \sqrt{5}/2$.

S93B17 The points are on the line $y = 2x + 1$ and in the interior of the circle $x^2 + y^2 = 25$. It is easy to find the four points (1,3), (0,1), (-1,-1) and (-2,-3).

S93B18 Since dividing by q leaves the same remainder, q must divide the difference between any two numbers.

$1594 - 1157 = 437 = 19 \cdot 23$

$1157 - 796 = 361 = 19 \cdot 19$

Therefore, $q = 19$, and simple division shows $r = 17$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS

SENIOR B DIVISION CONTEST NUMBER FOUR SPRING, 1993

S93B19 Adding the four equations, we obtain 5(a+b+c+d)=150 or a+b+c+d=30. Subtracting from each equation yields (2,5,9,14).

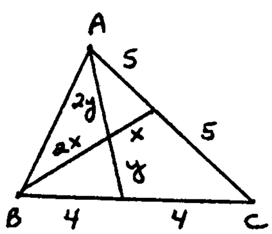
S93B20 $5^3 = 125 \equiv 6$ mod7, that is, yields a remainder of 6 when divided by 7. $5^6 = (5^3)^2 \equiv 6^2 \equiv 36 \equiv 1$ mod 7. Therfore, 5 to the power of any multiple of 6 yields a remainder of 1. Since 1993 is one more than a multiple of 6, the remainder is 1(5)=5.

S93B21 Let x represent the number of shots taken before last game $.375x + 11 = .4(x+12)$ which yeilds $x = 248$ and $x+12 = 260$.

S93B22 $\sqrt[3]{x} = 5/(4 + \sqrt[3]{x})$. Thus, $x^{2/3} + 4x^{1/3} - 5 = 0$
$(x^{1/3} + 5)(x^{1/3} - 1) = 0$ or $x = -125$ or $x = 1$.

S93B23 Using the formula $1 + 2 + \ldots + n = n(n+1)/2$, the sum of the first 133 integers is 8911. Therefore, 89 was counted twice to reach exactly 9000.

S93B24 Since the medians intersect at a point 2/3 of the way to the opposite side, we can set up the equations
$4x^2 + y^2 = 16$ and $x^2 + 4y^2 = 25$. Adding, we obtain
$5(x^2 + y^2) = 41$. Thus, $4(x^2 + y^2) = 164/5$. Since
$AB = \sqrt{4x^2 + 4y^2}$, $AB = \sqrt{164/5}$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER FIVE SPRING, 1993

S93B25 $(1+i)^2 = 1 + 2i - 1 = 2i$. $(1+i)^4 = (2i)^2 = -4$.
$(1+i)^{12} = (-4)^3 = -64$.

S93B26 Let x = the number of blue marbles. Then
$(x/25)((x-1)/24)((x-2)/23) = 143/1150 = 11\cdot13/23\cdot50$. Cross multiplying, $x(x-1)(x-2) = 11\cdot12\cdot13$. Therefore, $x=13$.

S93B27 Each side is $8 + 8\sqrt{3}$. Hence, the perimeter is $24 + 24\sqrt{3}$.

S93B28 Let $\log_5 72 = x$. $5^x = 72$. $x\log_{10}5 = \log_{10}72$.
$x = (\log_{10}72)/(\log_{10}5) = (\log_{10}9\cdot8)/(\log_{10}(10/2)) =$
$(2\log_{10}3 + 3\log_{10}2)/(\log_{10}10 - \log_{10}2) = (2q+3p)/(1-p)$.

S93B29 Alternately allowing none, 1, 2, 3, 4, and all the numbers to be negative, the values are 6, 2, -2, and -6.

S93B30 Using the law of cosines twice we obtain,
$6^2 = 4^2 + m^2 - 2\cdot4\cdot m\cdot\cos x$ and
$8^2 = 4^2 + m^2 - 2\cdot4\cdot m\cdot\cos(180 - x)$ or
$8^2 = 4^2 + m^2 + 2\cdot4\cdot m\cdot\cos x$

Adding, $6^2 + 8^2 = 4^2 + 4^2 + 2m^2$

$68 = 2m^2$

$m = \sqrt{34}$

$AC = 2m = 2\sqrt{34}$