NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE

PART I: TIME: 10 MINUTES SPRING, 1993

S93S1 Let $f(1)=1$, $f(2)=2$, $f(3)=3$ and $f(x+1) = (f(x-2)f(x-1) + 1)/f(x)$ for $x \geq 3$. Compute the value of $f(6)$.

S93S2 How many integers N are there such that $|N| \leq 1992$ and the product of the four consecutive integers beginning with N is a perfect square?

PART II: TIME: 10 MINUTES SPRING, 1993

S93S3 Compute the number of ordered pairs of integers (x,y) that satisfy $|x| + |y| < 5$.

S93S4 Compute the number of ordered pairs of integers (m,n), where $m>0$ and $n>0$, such that the region bounded by the lines $mx + ny = 1992$, $x = 0$, and $y = 0$ has an area of 2.

PART III: TIME: 10 MINUTES SPRING, 1993

S93S5 Compute the numerical value of $\cos x$ if $4^{\tan x} = 8^{\sin x}$ and $0<x<\pi$.

S93S6 Given that $abc = 3$, $a^3 + b^3 + c^3 = 7$ and $a^6 + b^6 + c^6 = 37$, find the value of $((a/b) + (b/c) + (c/a))((b/a) + (c/b) + (a/c))$.

<u>SOLUTIONS</u>

1. 4/7	3. 41	5. 2/3
2. 4	4. 45	6. 6

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO

PART I: TIME: 10 MINUTES SPRING, 1993

S93S7 The expression $\sqrt{6+3\sqrt{3}} - \sqrt{6-3\sqrt{3}}$ can be simplified to $\sqrt{m}$. Compute the value of m.

S93S8 Suppose that $\sum_{k=0}^{\infty} (1/a)^k + \sum_{k=0}^{\infty} (1/b)^k = \sum_{k=0}^{\infty} (1/ab)^k$ where $a \neq b$ and $|a| > 1$, $|b| > 1$. What is the value of $ab + 1/a + 1/b$?

PART II: TIME: 10 MINUTES SPRING, 1993

S93S9 A card is randomly drawn from a standard deck of 52 cards. What is the probability that the card is either a king, black, or a face card (jack, queen, or king) which isn't a spade?

S93S10 Find all ordered pairs of real numbers (a,b) such that $a/(a^2 + b^2) = 1$ and $b/(a^2 + b^2) = 3$.

PART III: TIME: 10 MINUTES SPRING, 1993

S93S11 Compute the real value of x that satisfies $\log_2 x + \log_4 x + \log_8 x = 11$.

S93S12 Compute $\log_2 i[2 + (1+i)^2 + (1+i)^3 + \ldots + (1+i)^{1991}]$.

SOLUTIONS

7. 6
8. 3
9. 8/13
10. (1/10, 3/10)
11. 64
12. 996

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE

PART I: TIME: 10 MINUTES SPRING, 1993

S93S13 Solve for x: $\sqrt[3]{x^2} + \sqrt[3]{x} - 20 = 0$.

S93S14 $F(x) = a|x-1| + b|x+1|$ with $b>a>0$. The area of the region bounded by $F(x)$, $y = 0$, $x = \pm 2$ is 100. Find $F(0)$.

PART II: TIME: 10 MINUTES SPRING, 1993

S93S15 If A, B, and C are positive integers less than ten, then find all such triplets (A,B,C) that satisfy $4!A + 5!B + 6!C = 1992$.

S93S16 In a race with constant speeds for each participant. Al beat Bob by 20 miles, Bob beat Carl by 10 miles, and Al beat Carl by 28 miles. How long was the race in miles?

PART III: TIME: 10 MINUTES SPRING, 1993

S93S17 Compute the numerical value of tan(arctan3 - arctan2).

S93S18 In triangle PQR, the ratio PR:RQ is 3:7. The bisector of the exterior angle at R intersects PQ extended at A. Compute the ratio AP:PQ. (P is between A and Q.)

SOLUTIONS

13. 64, -125 15. (3,4,2), (8,9,1), (8,3,2) 17. 1/7

14. 20 16. 100 miles 18. 3:4

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR

PART I: TIME: 10 MINUTES SPRING, 1993

S93S19 Find all ordered pairs (x,y) such that $|x| < 100$, $|y| < 100$ and

$(x - y + 9)^2 + (x + y - 1)^2 = 0$.

S93S20 In a sequence of complex numbers $(i=\sqrt{-1})$, suppose $T_0=1992$ and

$T_{n+1} + iT_n = 1$, compute $T_0 + T_1 + T_2 + \ldots + T_{1992}$.

PART II: TIME: 10 MINUTES SPRING, 1993

S93S21 The sum of the squares of the first and fourth terms of an arithmetic sequence is 410 and the sum of the squares of the second and third terms is 346. Compute the product of these four terms.

S93S22 How many positive divisors does $\left(\sum_{k=1}^{1992} k\right)^2 - \left(\sum_{k=1}^{1991} k\right)^2$ have?

PART III: TIME: 10 MINUTES SPRING, 1993

S93S23 Compute the sum of the numerical coefficients in the expansion of $(2x - y)^6$.

S93S24 How many positive integers satisfy the following:
(i) has seven digits, (ii) is divisible by eleven, and (iii) the sum of its digits is 59.

SOLUTIONS

19. $(-4, 5)$
20. $2988 - 996i$
21. 21,945
22. 160
23. 1
24. 40

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE

PART I: TIME: 10 MINUTES SPRING, 1993

S93S25 $100!/3^k$ is an integer, where k is an integer. Compute the largest value of k.

S93S26 In square ABCD with AB=1, K is between B and C. E is on the extension of AK with K between A and E. $F = \overleftrightarrow{EC} \cap \overleftrightarrow{AD}$. If the area of triangle AEF = 3 and AE:AK = 7, compute BK.

PART II: TIME: 10 MINUTES SPRING, 1993

S93S27 Compute the sum of the reciprocals of the roots of $x^3 - 5x^2 + 8x - 6 = 0$

S93S28 Let $F(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$. Compute $F(999)/10^6$.

PART III: TIME: 10 MINUTES SPRING, 1993

S93S29 Circle O is inscribed in quadrilateral ABCD. If AB=2, BC=4 and CD=18, find the length of AD.

S93S30 Suppose that for all points on $x^4 = kx^6 + y^4$, the maximum value of $\sqrt{x^2 + y^2} + \sqrt{x^2 - y^2}$ is 1/17 where k>0. Compute the value of k.

SOLUTIONS

25. 48
26. 13/49
27. 4/3
28. 998,002
29. 16
30. 1156

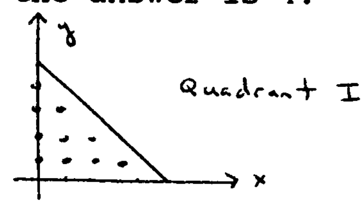
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER ONE SPRING, 1993

S93S1 $f(4)=(f(1)f(2) + 1)/f(3) = 3/3 = 1$
$f(5)=(f(2)f(3) + 1)/f(4) = 7/1 = 7$
$f(6)=(f(3)f(4) + 1)/f(5) = 4/7$

S93S2 Look at $n(n+1)(n+2)(n+3) = n^4 + 6n^3 + 11n^2 + 6n = (n^2 + 3n + 1)^2 - 1$. Thus, $n^2 + 3n + 1 = \pm 1$ which corresponds to $n(n+1)(n+2)(n+3) = 0$. Thus, the answer is 4.

S93S3 We see that there are $4(1 + 2 + 3 + 4) + 1 = 41$ solutions.



S93S4 The x and y intercepts are $1992/m$ and $1992/n$. The area is $(1/2)(1992^2/mn) = 2$ or $mn = (1/4)(2^3 \cdot 3 \cdot 83)^2 = 2^4 \cdot 3^2 \cdot 83^2$. Now $2^4 \cdot 3^2 \cdot 83^2$ has $(4+1)(2+1)(2+1)$ positive divisors; thus the answer is 45.

S93S5 $2^{2\tan x} = 2^{3\sin x}$ which yields $2(\sin x/\cos x) = 3\sin x$. Since $\sin x \neq 0$, we obtain $2/\cos x = 3$ or $\cos x = 2/3$.

S93S6 Multiplying yields $3 + \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2} + \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab}$

$= 3 + (1/a^2b^2c^2)[b^3c^3 + a^3c^3 + a^3b^3 + a^4bc + b^4ac + c^4ab]$

$= 3 + (1/a^2b^2c^2)[a^3b^3 + b^3c^3 + c^3a^3 + abc(a^3 + b^3 + c^3)]$

$= 3 + (1/a^2b^2c^2)[\tfrac{1}{2}[(a^3 + b^3 + c^3)^2 - (a^6 + b^6 + c^6)] + abc(a^3 + b^3 + c^3)]$

$= 3 + (1/3^2)[\tfrac{1}{2}(7^2 - 37) + 3 \cdot 7]$

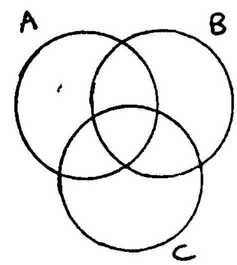
$= 3 + (1/9)[6 + 21] = 6$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER TWO SPRING, 1993

S93S7 Let $a=3\sqrt{3}$ and $x = \sqrt{6+a} - \sqrt{6-a}$.
$x^2 = 6 + a - 2\sqrt{36 - a^2} + 6 - a = 12 - 2\sqrt{36 - a^2}$.
$x = \sqrt{12 - 2\sqrt{36 - a^2}} = \sqrt{12 - 2\sqrt{36 - 27}} = \sqrt{12 - 2\sqrt{9}} = \sqrt{6}$.
$m = 6$

S93S8 These are geometric series, so we have
$1/(1 - 1/a) + 1/(1 - 1/b) = 1/(1 - 1/ab)$
$a/(a-1) + b/(b-1) = ab/(ab - 1)$ or $(2ab - a - b)/(a-1)(b-1)$
$= ab/(ab-1)$. Thus, $ab(a-1)(b-1) = (ab-1)(2ab - a - b)$.
$a^2b^2 - a^2b - ab^2 + ab$
$= 2a^2b^2 - a^2b - ab^2 - 2ab + a + b$
$a^2b^2 + a + b = 3ab$
Thus, $ab + 1/b + 1/a = 3$.

S93S9 Consider the diagram where A,B,C are collections of items. Let N(S) be the size of S. Then by looking at the areas we see that $N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(B \cap C) - N(C \cap A) + N(A \cap B \cap C)$. Let A=king, B=black, and C= face card not a spade. Then the favorable number of cards is $4 + 26 + 9 - 2 - 3 - 3 + 1 = 32$. Therefore the answer is $32/52 = 8/13$.



S93S10 Dividing the expressions yields $a/b = 1/3$ or $b = 3a$.
Substituting yields $3a/(a^2 + 9a^2) = 3a/10a^2 = 3$ or
$10a^2 - a = 0$. Therefore, $a=0$ or $1/10$. $a=0$ yields no solution, therefore, $a = 1/10$ and $b=3/10$. The answer is $(1/10, 3/10)$.

S93S11 We know $(\log_a b)(\log_b c) = \log_a c$. Therefore,
$\log_2 x + (\log_4 2)(\log_2 x) + (\log_8 2)(\log_2 x) = 11$
$\log_2 x + (1/2)\log_2 x + (1/3)\log_2 x = 11$
$(\log_2 x)[1 + 1/2 + 1/3] = 11$
$(11/6)\log_2 x = 11$. Hence, $\log_2 x = 6$ or $x = 2^6 = 64$.

S93S12 $(1+i)^2 + \ldots + (1+i)^{1991} =$
$(1+i)^2[(1+i)^{1990} - 1]/[(1+i) - 1] = 2i[(2i)^{995} - 1]/i =$
$2[2^{995}i^{995} - 1] = 2^{996}(-i) - 2$. We seek
$\log_2 i[2 + 2^{996}(-i) - 2] = \log_2 2^{996} = 996$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

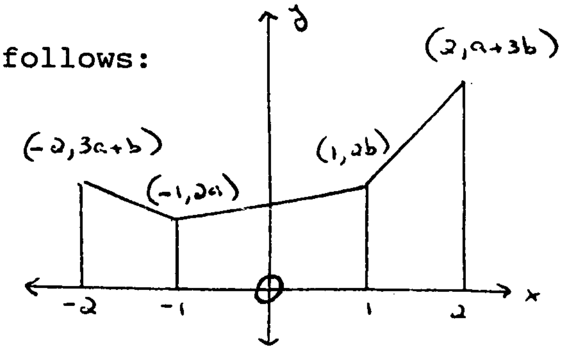
SOLUTIONS CONTEST NUMBER THREE SPRING, 1993

S93S13 $x^{2/3} + x^{1/3} - 20 = 0$. Let $a = x^{1/3}$. This yields $a^2 + a - 20 = 0$ or $(a+5)(a-4) = 0$. Thus, $a = -5$ or $a = 4$. Therefore, $x^{1/3} = -5$ or $x = -125$ and $x^{1/3} = 4$ or $x = 64$. Upon checking, both -125 and 64 are solutions.

S93S14 F(x) can be expressed piecewise as follows:

$$F(x) = \begin{cases} (a+b)x + (b-a), & x \geq 1 \\ (b-a)x + (a+b), & -1 \leq x \leq 1 \\ -(a+b)x + (a-b), & x \leq -1 \end{cases}$$

Area is $((2-1)/2)[2b + a + 3b] + ((1-(-1))/2)[2a + 2b] + ((-1-(-2))2)[3a + b + 2a]$
$= \frac{1}{2}(a + 5b) + 2a + 2b + \frac{1}{2}(5a + b)$
$= 5(a + b)$. Therefore, $a + b = 20$, thus $F(0) = a + b = 20$.

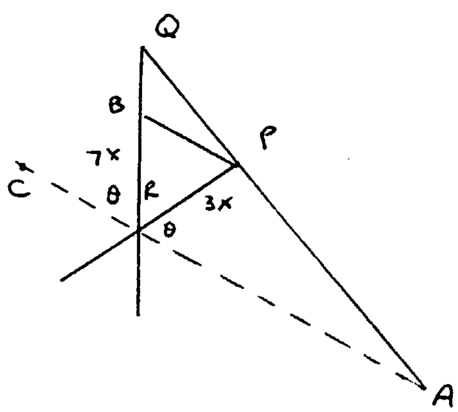


S93S15 We have $A + 5B + 30C = 83$. Thus, $A \equiv 3 \bmod 5$ which yields $A = 3$ or 8. If $A=3$, then $B + 6C = 16$ which yields $C=2$ and $B=4$. If $A=8$, then $B + 6C = 15$ which yields $(C,B) = (1,9)$ or $(2,3)$. Hence the answers are (3,4,2), (8,9,1), and (8,3,2).

S93S16 Let the race have distance D and R_A, R_B, and R_C represent the rates of Al, Bob, and Carl. Now $D/R_A = (D-20)/R_B = (D-28)/R_C$ and $D/R_B = (D-10)/R_C$. Now $R_B/R_C = (D-20)/(D-28) = D/(D-10)$. Thus, $D^2 - 28D = D^2 - 30D + 200$ or $2D = 200$ or $D = 100$ miles.

S93S17 Let $A = \arctan 3$ and $B = \arctan 2$. Then, $\tan(A-B) = (\tan A - \tan B)/(1 + \tan A \tan B) = (3-2)/(1 + 3\cdot 2) = 1/7$.

S93S18 Let PR=3x and RQ=7x. Draw PB parallel to AR. Since angle PRA equals angle QRC, we have RB=RP=3x. Thus, QB=4x. Now QP/QA = QB/QR, we obtain QP/(QP+PA) = 4x/7x = 4/7. Now (QP+PA)/QP = AP/PQ + 1 = 7/4 or AP/PQ = 3/4. Thus, AP:PQ = 3:4.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SOLUTIONS CONTEST NUMBER FOUR SPRING, 1993

S93S19 Since $(x - y + 9)^2 \geq 0$ and $(x + y - 1)^2 \geq 0$, the only possibility is $x - y + 9 = 0$ and $x + y - 1 = 0$. Solving gives $x = -4$ and $y = 5$. Therefore the solution is $(-4, 5)$.

S93S20 Notice $T_1 = 1 - iT_o$, $T_2 = 1 - i - T_o$, $T_3 = 1 - i + i^2 + iT_o = i(T_o - 1)$, $T_4 = 1 - i^2(T_o - 1) = T_o$. Thus, $T_k = T_{k+4}$. Now $T_1 + T_2 + T_3 + T_4 = 2 - 2i$. $\sum_{k=1}^{1992} T_k = (1992/4)\cdot(2 - 2i) = 996 - 996i$. Now add T_o and the answer is $2988 - 996i$.

S93S21 Let the terms of the arithmetic sequence be represented as $a-3d$, $a-d$, $a+d$, and $a+3d$. We know $(a-3d)^2 + (a+3d)^2 = 410$ and $(a-d)^2 + (a+d)^2 = 346$. Simplifying yields $2a^2 + 18d^2 = 410$ and $2a^2 + 2d^2 = 346$. Therefore, $d^2 = 4$ and $a^2 = 169$. The product is $(a^2 - 9d^2)(a^2 - d^2) = (133)(165) = 21{,}945$.

S93S22 The difference equals $\left(\sum_{k=1}^{1992} k - \sum_{k=1}^{1991} k\right)\left(\sum_{k=1}^{1992} k + \sum_{k=1}^{1991} k\right) = 1992[1992 + 2\sum_{k=1}^{1991} k] = 1992[1992 + 1992\cdot 1991] = 1992[1992^2] = 1992^3 = (2^3\cdot 3\cdot 83)^3 = 2^9\cdot 3^3\cdot 83^3$. Thus, $(9+1)(3+1)(3+1) = 160$.

S93S23 Letting $x=y=1$, we obtain the sum of the coefficients. Therefore, $(2 - 1)^6 = 1$.

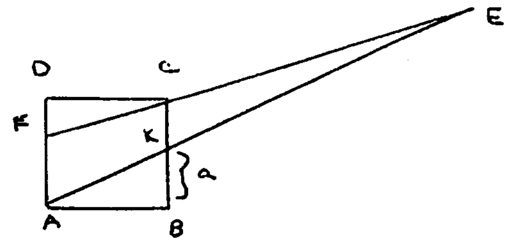
S93S24 Let the numbers be represented as $x_1y_1x_2y_2x_3y_3x_4$, where x_i and y_i are digits. To satisfy (ii) we must have 11 dividing $|\Sigma x_i - \Sigma y_i|$ or $\Sigma x_i - \Sigma y_i = 0, \pm 11, \pm 22, \pm 33$ since $1 \leq \Sigma x_i \leq 36$ and $0 \leq \Sigma y_i \leq 27$. Also $\Sigma x_i + \Sigma y_i = 59$. Thus $(\Sigma x_i, \Sigma y_i) = (35,24)$ is the only permissible solution. Now $\Sigma x_i = 35$ means that (x_1,x_2,x_3,x_4) is an arrangement of $(9,9,9,8)$, 4 possibilities. Now (y_1,y_2,y_3) is an arrangement of one $(9,9,6)$, $(9,8,7)$ or $(8,8,8)$; a total of $3 + 6 + 1 = 10$ possibilities. Thus, the answer is $4\cdot 10 = 40$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SOLUTIONS CONTEST NUMBER FIVE SPRING, 1993

S93S25 Applying the Greatest Integer Function, we obtain
$[100/3] + [100/9] + [100/27] + [100/81] = 33 + 11 + 3 + 1 = 48$.

S93S26 Let $BK=a$. Then $CK = 1-a$. Now $EK/EA = 1 - AK/AE = 1 - 1/7 = 6/7$. By similarity $EK/EA = 6/7 = CK/FA = (1-a)/FA$ or $FA = (7/6)(1-a)$. Now with FA as base the height of triangle FAE is 7. Thus, $3 = \frac{1}{2}(7)(7/6)(1-a)$ or $a = 13/49$.



S93S27 Let a, b, and c represent the three roots. Then the sum of the reciprocals is $1/a + 1/b + 1/c = (bc + ac + ab)/abc$. Therefore, $bc + ac + ab = 8$ and $abc = 6$. Hence,
$1/a + 1/b + 1/c = 8/6 = 4/3$

S93S28 The answer follows fairly easy by finding $F(10^3 - 1)$ using the Binomial Theorem. But a better solution uses the following;
$F(x) = x^4 + 4x^3 + 6x^2 + 4x + 1 - 2x^3 - 4x^2 - 2x$
$= (x+1)^4 - 2x(x^2 + 2x + 1) = (x+1)^2[(x+1)^2 - 2x]$
$= (x+1)^2(x^2+1)$. Now
$F(10^3 - 1) = (10^3)^2((10^3-1)^2 + 1)$
$= 10^6[10^6 - 2\cdot10^3 + 1 + 1] = 10^6[998002]$. Thus,
$F(999)/10^6 = 998002$.

S93S29 Since the opposite sides of a circumscribed quadrilateral sum identically, we have $AB + CD = BC + AD$. Thus, $AD = 2+18-4 = 16$.

S93S30 First $y^4 = x^4(1-kx^2)$. Thus, we must have $1-kx^2 \geq 0$ or $(-1/\sqrt{k}) \leq x \leq (1/\sqrt{k})$. Let $T = \sqrt{x^2 + y^2} + \sqrt{x^2 - y^2}$, then $T^2 = 2(x^2 + \sqrt{x^4 - y^4})$
$= 2(x^2 + \sqrt{(kx^6 + y^4) - y^4}) = 2(x^2 + \sqrt{k}x^3)$. Now
T^2 increases as x gets larger, thus $x = 1/\sqrt{k}$ and $T^2 \leq 2((1/k) + \sqrt{k}(1/k\sqrt{k})) = 4/k$. Now $T \leq 2/\sqrt{k}$ and the maximum value of $\sqrt{x^2 + y^2} + \sqrt{x^2 - y^2} = 2/\sqrt{k}$ or $1/17 = 2/\sqrt{k}$.
Thus, $\sqrt{k} = 34$ and $k = 1156$.