NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — SPRING 1993

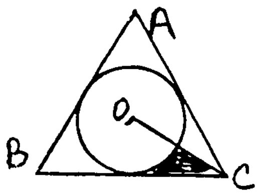
Part I: 10 Minutes

S93J1. *The altitude to the hypotenuse of a right triangle has length $3\sqrt{3}$. If the longer segment on the hypotenuse has length 9, compute the length of the shorter leg of the given triangle.*

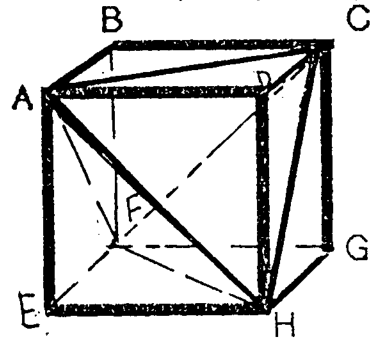
S93J2. $\dfrac{1992! + 1993!}{1994!}$ *can be reduced to a/b where a and b are relatively prime integers. Write the ordered pair (a,b).*

Part II: 10 Minutes — NYCIML CONTEST ONE — SPRING 1993

S93J3. *Each side of equilateral $\triangle ABC$ has length $\sqrt{12}$. Circle O is inscribed in $\triangle ABC$. The shaded area can be written in simplest form as $\dfrac{a\sqrt{b}+c}{6}$. Write the ordered triple (a,b,c).*



S93J4. *Compute the ratio of the surface area of tetrahedron FAHC to the surface area of cube ABCDEFGH.*



Part III: 10 Minutes — NYCIML CONTEST ONE — SPRING 1993

S93J5. *How many positive integers smaller than 100 are relatively prime to 100?*

S93J6. *Player A can beat player B half the time. Player A can beat player C 1/3 of the time. Player A must alternate games with players B and C and must decide on his first opponent by tossing a coin. Up to three games can be played. If player A is considered the winner after winning 2 games in a row, compute the probability that player A wins.*

Answers

1. 6
2. (1,1993)
3. $(3,3,-\pi)$
4. $\sqrt{3}:3$ (or equivalent)
5. 40
6. 19/72

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE		
JUNIOR DIVISION	CONTEST NUMBER TWO	SPRING 1993

Part I: 10 Minutes

S93J7. *The diagonals of quadrilateral ABCD are perpendicular and meet at point E. If $CE = EB = 2\sqrt{6}$, $ED = 2\sqrt{2}$, find the ratio of $m\angle EDC$ to $m\angle EBC$.*

S93J8. *Compute the number of positive integers smaller than 1600 that are also relatively prime to 1600.*

Part II: 10 Minutes NYCIML CONTEST TWO SPRING 1993

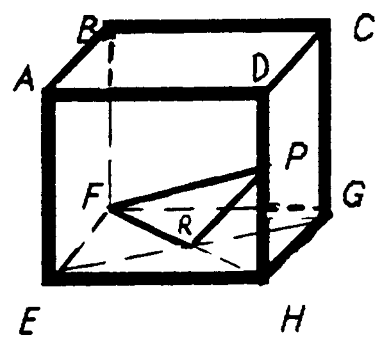
S93J9. $\dfrac{1993! - 1992!}{1994! - 2(1992!)}$ *can be reduced to a/b where a and b are relatively prime. Find the ordered pair (a,b).*

S93J10. *Compute the value of*

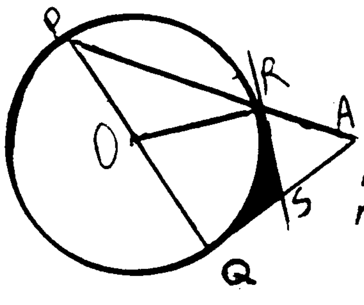
$$\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{50}\right)$$

Part III: 10 Minutes NYCIML CONTEST TWO SPRING 1993

S93J11. *Let P be the midpoint of edge $\overline{DH}$ in unit cube ABCDEFGH. Let R be the intersection of $\overline{FH}$ and $\overline{EG}$. Find the perimeter of $\triangle PFR$.*



S93J12.



In isosceles $\triangle PQA$, $\overline{PQ}$ is a diameter of circle O and PQ=2. $\overline{RS}$ and $\overline{QS}$ are tangent to circle O at R and Q respectively. Find the shaded area.

Answers

7. 4: 3 or equivalent
8. 640
9. (1, 1995)
10. 1/50 or .02
11. $\dfrac{3+\sqrt{3}+\sqrt{2}}{2}$
12. $\dfrac{4-\pi}{4}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER THREE — SPRING 1993

Part I: 10 Minutes

S93J13. *Compute the value of n if*

$$\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)=0.002$$

S93J14. *Suppose that a side of square ABCD is 8 inches long. If E is the midpoint of side $\overline{AB}$ and $\overline{EF} \perp \overline{DE}$, find the number of inches in the length of $\overline{FB}$. (F is on $\overline{BC}$.)*

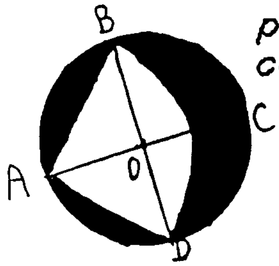
Part II: 10 Minutes — NYCIML CONTEST NUMBER THREE — SPRING 1993

S93J15. *Recall that $\phi(N)$ is the number of positive integers less than or equal to N that are relatively prime to N. Find the sum $\phi(d_1)+\phi(d_2)+\phi(d_3)+\ldots+\phi(d_k)$ where $d_1, d_2, d_3, \ldots, d_k$ are the divisors of 7200.*

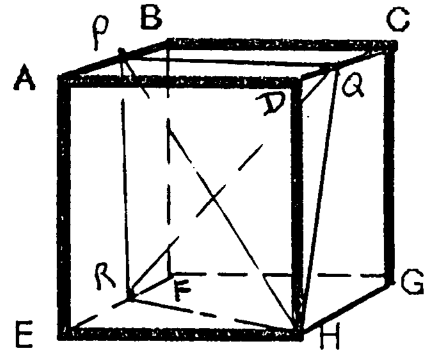
S93J16. $\dfrac{3982(1991!)+2(1991!)}{1991(1992!)+1993!}$ *can be reduced to a/b where a and b are relatively prime integers. Find the ordered pair (a,b).*

Part III: 10 Minutes — NYCIML CONTEST NUMBER THREE — SPRING 1993

S93J17. *In circle O, diameters $\overline{AC}$ and $\overline{BD}$ are $\perp$. An arc of circle A passes through B and D. If the shaded area is 32π, compute the length of a radius of circle O.*



S93J18. *If P, Q, and R are midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ of unit cube ABCDEFGH, find the surface area of tetrahedron PQHR.*



Answers

13. 1000

14. 2

15. 7200

16. (1, 1992)

17. 8

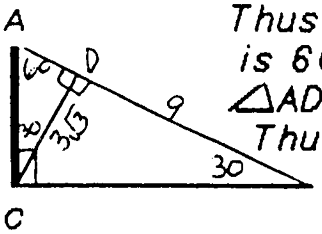
18. $\frac{1}{2}+\frac{\sqrt{5}}{2}+\frac{\sqrt{6}}{4}$ or equivalent

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 1993**

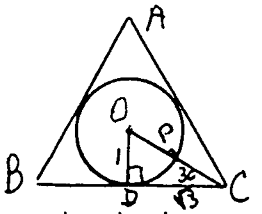
SOLUTIONS

S93J1. *Note that $BD = CD\sqrt{3}$. Thus $m\angle B = 30$ and $m\angle A$ is 60. This means $\triangle ADC$ is 30-60-90. Thus,* $\boxed{AC = 6}$



S93J2. *The given expression is equal to* $\dfrac{1992!(1+1993)}{1994\cdot1993\cdot1992!}$ *Thujs we can divide by the common factor 1992! giving* $\dfrac{1994}{(1994)(1993)} = \boxed{\dfrac{1}{1993}}$

S93J3. *Note that $\sqrt{12} = 2\sqrt{3}$. This means that $DC = \sqrt{3}$ and Area of $\triangle ODC = \sqrt{3}/2$. The area of sector $OPD = \pi/6$ so the shaded area is the difference* $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6} = \dfrac{3\sqrt{3}-\pi}{6}$ *Answer: $(3,3,-\pi)$*



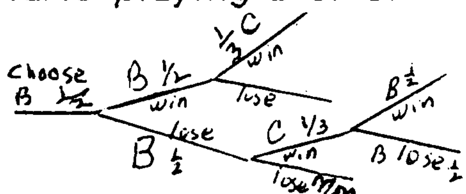
S93J4. *Let a side of the cube have length one. Then diagonals $AC = FH = CH = HA = \sqrt{2}$. Thus each face of the tetrahedron is an equilateral triangle with area $\sqrt{3}/2$. The surface area of the tetraheron is thus $2\sqrt{3}$. The surface area of the cube is 6. The ratio is therefore $2\sqrt{3} : 6$ or* $\boxed{\sqrt{3} : 3}$

S93J5. *Although many students may actually count the number of positive integers less than 100, relatively prime to 100 to get the answer 40, an easier approach is to use Euler's (pronounced "Oiler's) ϕ function: If $n = p_1^{a} \cdot p_2^{b}$,*

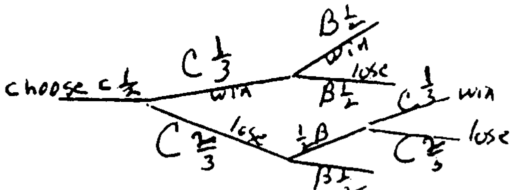
$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$$

(This can be extended to any number of factors in the PRIME factorization of n.) Since $100 = 2^2 \cdot 5^2$, the formula gives $100(1/2)(4/5) =$ $\boxed{40}$

S93J6. *The following diagrams show the two possible tree diagrams for this problem, depending upon whether player A starts playing B or C.*



P(win) $= \dfrac{1}{2}\cdot\dfrac{1}{6} + \dfrac{1}{2}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{8}$



P(win) $= \dfrac{1}{2}\cdot\dfrac{1}{3}\cdot\dfrac{1}{2} + \dfrac{1}{2}\cdot\dfrac{2}{3}\cdot\dfrac{1}{2}\cdot\dfrac{1}{3} = \dfrac{5}{36}$

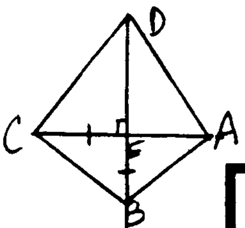
Thus the probability of a win is found by adding the above two fractions giving: $\boxed{\dfrac{19}{72}}$

New York City Interscholastic Mathematics League

JUNIOR DIVISION CONTEST NUMBER TWO SPRING 1993

SOLUTIONS

S93J7. *It is obvious that $\triangle BCE$ is 45-45-90. Since $EC = ED\sqrt{3}$, $\triangle CDE$ is 30-60-90. Thus $m\angle EDC = 60$ and $m\angle EBC = 45$, giving the desired ratio as*



$$\boxed{60 : 45 = 4 : 3}$$

S93J8. *The prime factorization of 1600 is*

$$1600 = 2^6 \cdot 5^2.$$

Using Euler's ϕ function (See S93J5) we get

$$\phi(1600) = 1600\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 1600(1/2)(4/5) = \boxed{640}$$

S93J9. *Divide the numerator and denominator of the given expression by common factor 1992 ! This gives:*

$$\frac{1993 - 1}{1994(1993) - 2} = \frac{1993 - 1}{(1993+1)1993 - 2}$$

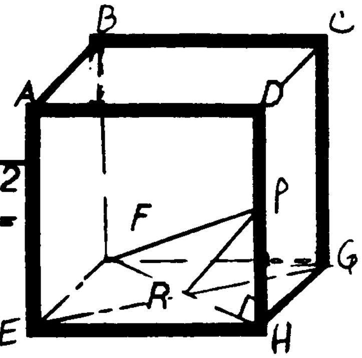
$$= \frac{1993 - 1}{1993^2 + 1993 - 2}$$

$$= \frac{(1993 - 1)}{(1993+2)(1993-1)} = \boxed{\frac{1}{1995}}$$

S93J10. *The given example leads to the following "telescoping" product:*

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{49}{50} = \boxed{\frac{1}{50}}$$

S93J11.



A diagonal of any face has length $\sqrt{2}$ meaning $FR = RH = \sqrt{2}/2$. Since $PH = 1/2$, we can use the Pythagorean Theorem for $\triangle HRP$:

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = PR^2$$

This gives $PR = \sqrt{3}/2$. Use the Pythagorean Theorem again, this time with $\triangle PHF$, we get

$$FH^2 + PH^2 = PF^2, \text{ or}$$

$$\sqrt{2}^2 + (1/2)^2 = PF^2$$

So $PF = 3/2$. This means the perimeter is

$$\boxed{\frac{3 + \sqrt{3} + \sqrt{2}}{2}}$$

S93J12. *The diagram was NOT drawn to scale, since it would be a dead giveaway that QORS is a square! Since a diameter is perpendicular to a tangent at the contact point, $m\angle PQS = 90$ and $\triangle PQA$ is 45-45-90. Since $\triangle OPR$ is isosceles, $m\angle ORP = m\angle OPR = 45$. This also gives $m\angle SRA = 45$. Thus $\triangle RAS$ is 45-45-90 and $m\angle ORS = 90$. These give us the fact that ORSQ is a square whose side has length 1 and whose area is 1. The area of sector QOR is $\pi/4$. Thus the shaded area is $1 - \pi/4$*

$$= \boxed{\frac{4 - \pi}{4}}$$

New York City Interscholastic Mathematics League

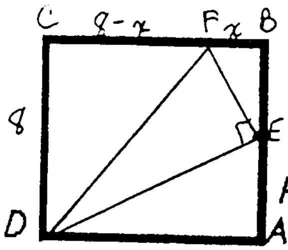
JUNIOR DIVISION   CONTEST NUMBER THREE   SPRING 1993

SOLUTIONS

S93J13. The given product "telescopes" to $\frac{2}{n} = .002$

This yields $\boxed{n = 1000}$

S93J14. Let $x = FB$; $BE = 4$



and $FE = \sqrt{16 + x^2}$. $DE = \sqrt{80}$. To find DF, we use the Pythagorean Theorem twice, once with $\triangle FCD$ and once with $\triangle DEF$. This gives:

$$64 + (8 - x)^2 = 80 + 16 + x^2$$

This gives $\boxed{x = 2.}$

S93J15. The easiest way to do this problem is to take test values and generalize:
Example, If $N = 6$, the d's are 1, 2, 3, and 6. $\phi(1) + \phi(2) + \phi(3) + \phi(6) = 1 + 1 + 2 + 2 = 6$.
If $N = 10$, the d's are 1, 2, 5, and 10. $\phi(1) + \phi(2) + \phi(5) + \phi(10) = 1 + 1 + 4 + 4 = 10$
A few more examples will convince you that the sum of all of $\phi(d)$'s will always be N. Thus for 7200,

this sum is $\boxed{7200.}$

S93J16. Dividing numerator and denominator by 1991 ! gives

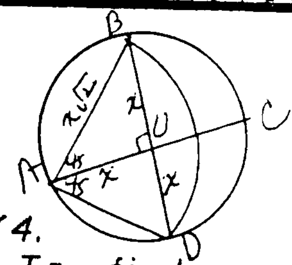
$$\frac{2(1991) + 2}{1991(1992) + 1993(1992)}$$

$$= \frac{2(1991 + 1)}{1991(1991+1) + 1993(1991+1)}$$

$$= \frac{2}{1991 + 1993} = \frac{1}{1992}$$

$\boxed{\text{Thus the ordered pair is } (1, 1992)}$

S93J17.



A(shaded)= A(of circle O) − (Area of Circle A)/4.
Let $OB = x$. To find x, use 45–45–90 $\triangle$ AOB:

$$A(\text{shaded}) = \pi x^2 - \pi x^2 2/4$$

$$= \frac{\pi}{2} x^2 = 32\pi$$

Solving for x yields $\boxed{x = 8}$

S93J18. The key to this problem is to find the lengths of the four edges and then the areas of the four triangles. It should be obvious that $PQ = PR = 1$. By the Pythagorean Theorem $RQ = \sqrt{2}$, $PH = 3/2$, and $RH = QH = \sqrt{5}/2$.
a) Since $\triangle PQR$ is a right triangle, its area is easily found to be 1/2.
b) $\triangle$ PRH is right and its area is easily found to be $\sqrt{5}/4$.
c) Since $\triangle RQH$ is isosceles with base $\sqrt{2}$ and altitude $\sqrt{3}/2$ its area is $\sqrt{6}/4$.
d) $\triangle PQH$ is a a right triangle, (but it is not so obvious) and its area is found to be $\sqrt{5}/4$.
Adding these areas we get a surface area of

$$\boxed{\frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{4}.}$$