NYCIML
JUNIOR DIVISION
SPRING 1992

CONTEST 1

PART 1: 10 MINUTES

S92J1 How many different ordered triplets of integers (x, y, z) have the property that $x^2 + y^2 + z^2 < 7$?

S92J2 Compute the numerical value of $\left[\dfrac{10}{\sqrt{10005}-\sqrt{10010}}\right]$. (Note that $[x]$ means the largest integer $\leq x$.

PART 2: 10 MINUTES

S92J3 If $P = \dfrac{A}{B}$, $Q = \dfrac{B}{C}$, $R = \dfrac{C}{A}$, express $\dfrac{A+B}{A+C}$ as a single simplified fraction in terms of P and R.

S92J4 Chords $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ of a circle are concurrent at interior point P. If AP=2, CP=4, EP=6 and AB+CD+EF=45, find DP.

PART 3: 10 MINUTES

S92J5 A sequence is defined by $a_1 = 74$: $a_n = (a_{n-1})/4$ if a_{n-1} is divisible by 4; $a_n = 5a_{n-1} + 22$ otherwise. Compute the value of a_{1990}.

S92J6 Square ABCD has A(0,0) and C(0,2). Equilateral $\triangle ABC$ (in the same plane) has E(2,0), EG on the x-axis with E between A and G. The y-coordinate of F is 2. Compute the minimum horizontal distance from a point on the square to a point on the triangle.

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CONTEST 2

PART 1: 10 MINUTES

S92J7 In right $\triangle ABC$, $\sin A + \sin B + \sin C = 2.39$. Compute the value of $\cos A + \cos B + \cos C$.

S92J8 Find the ordered pair of integers (x, y) such that $71x - 13y = 1$ and $10 \le x < 20$ and $10 \le y < 20$.

PART 2: 10 MINUTES

S92J9 On the sides of angle ACE, B is between A and C, D is between C and E, and EA=AD=DB=BC. F is on ray DE with E between D and F. If angle ACE= 21°, compute angle AEF.

S92J10 In trapezoid ABCD, bases AB=15, and CD=30. Points E and F are on AD and BC with $EF \parallel AB$. If the ratio of the area of ABFE to that of EFCD is 13:12, compute EF.

PART 3: 10 MINUTES

S92J11 In a circle, chord AB=2, chord AC=4, and angle BAC= 30°. A perpendicular from B to AC is extended until it intersects the circle again at D. Compute BD.

S92J12 In trapezoid ABCD, shorter leg AB, shorter base BC and longer leg CD are three consecutive integers in increasing order, respectively. Longer base DA is twice base BC. Find the two smallest positive integer values of base BC for which this can occur, if the height must also be an integer.

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CONTEST 3

PART 1: 10 MINUTES

S92J13 The standard form of a complex number is $x + yi$ with x and y real and $i^2 = -1$. If a + bi is the reciprocal of c + di, and $\frac{a}{b}$ is defined and equal to $\frac{c}{d}$, express (b + ai)(c + di) in simplest form.

F92J14 In $\triangle ABC$, $m\angle A = 80^\circ$ and $m\angle B = 70^\circ$. Point D is in the interior of $\triangle ABC$ with $\overline{AD} \cong \overline{AB} \cong \overline{BD}$. Compute $m\angle ADC$.

PART 2: 10 MINUTES

S92J15 Find the smallest integer value of x for which $x^2 - 30x - 175$ is equal to a positive prime number.

S92J16 Let the time 9:04:$\frac{5}{7}$ represent the number $904\frac{5}{7}$. (Likewise the time 12:34:$\frac{3}{4}$ represent $1234\frac{3}{4}$ and 12:34 represents 1234, etc.) Find all times which exactly represent the degree-measure of the smaller angle between the hour and the minute hand at those times.

PART 3: 10 MINUTES

S92J17 If n is a positive integer, find the three smallest values of n for which $\sqrt{\frac{n(n+1)}{2}}$ is also an integer.

S92J18 Four distinct points are chosen randomly on a circle and randomly labeled A, B, C, D (one letter to a point). Compute the probability that $m\angle ABC + m\angle BCD + m\angle CDA + m\angle DAB \geq 180^\circ$.

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CONTEST 1 - SOLUTIONS

S92J1 If z = 0 then $x^2 + y^2 < 7$. There are two values each on the positive and negative x, y, and z axes, one at the origin, and three in each quadrant for a total of 21. If z = ± 1, then $x^2 + y^2 < 6$, which is satisfied by the same exact values of x and y as before. If z = ± 2, then $x^2 + y^2 < 3$, giving 9 points. Thus, there are 21 + 2(21 + 9) = 31 triplets.

S92J2 The given expression can be rewritten as:

$$\left[\frac{10\left(\sqrt{10005}+\sqrt{10010}\right)}{10005-10010}\right]=\left[-2(100^{+}+100^{+})\right]=\left[-2(200^{+})\right]= - 401$$

S92J3 $$\frac{A+B}{A+C}=\frac{BP+B}{A+AR}=\frac{B(P+1)}{A(R+1)}=\frac{P+1}{P(R+1)}=\frac{P+1}{PR+P}$$

S92J4 By the chord-product theorem, we can label PF = 2x, PD = 3x and PB = 6x. From the given, we have 2 + 4 + 6 + 2x + 3x + 6x = 45. Thus, x = 3 and DP = 3(3) = 9.

S92J5

n	1	2	3	4	5	6	7	8	9
a_n	74	392	98	512	128	32	8	2	32

One can see that $a_{3k} = 32$ for $k \le 2$. Now $a_{3(663)} = a_{1989} = 32$. This implies that $a_{1990} = 8$.

S92J6 $$EH = \frac{2\sqrt{3}}{3}$$

$\frac{EQ}{PQ}=\frac{EH}{FH}$ giving $\frac{EQ}{1}=\frac{\frac{2\sqrt{3}}{3}}{2}$. This yields $EQ=\frac{\sqrt{3}}{3}$. The minimum distance is clearly equal to BP, which is equal to $1+\frac{\sqrt{3}}{3}$.

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CONTEST 2 - SOLUTIONS

S92J7 $\sin A + \sin B + \sin C = \frac{a}{c} + \frac{b}{c} + 1 = 2.39$

$\cos A + \cos B + \cos C = \frac{b}{c} + \frac{a}{c} + 0$

which is equal to 1.39.

S92J8 13y = 17x –1 $\rightarrow$ $y = x + \frac{4x-1}{13}$. Since x and y must be integers, $\frac{4x-1}{13}$ must also be an integer, call it t. $t = \frac{4x-1}{13}$ $\rightarrow$ $x = 3t + \frac{t+1}{4}$. Now $\frac{t+1}{4}$ must be an integer and t = 3 will make it an integer. This gives us 4x = 40 and x = 10, y = 13.

S92J9 Let x = m($\angle ACD$). Then $m(\angle AEF) = 180^\circ - 3x = 180^\circ - 3(21^\circ) = 117^\circ$.

S92J10 Extend $\overline{AD}$ and $\overline{BC}$ to meet at O. $\overline{AB}$ is a "midline" of triangle CDO so the area of $\triangle AOB$ = ¼ o the area of $\triangle CDO$. The area of $\triangle AOB$ = $\frac{1}{3}$ of the area of ABCD. Let 13x = the area of ABEF and 12x = the area of EFCD. Then the area of $\triangle AOB$ is $\frac{25x}{3}$. Notice that $\frac{Area\triangle EFO}{Area\triangle CDO} = \frac{EF^2}{CD^2}$. Thus, we have $\frac{EF^2}{CD^2} = \frac{13x + \frac{25x}{3}}{25x + \frac{25x}{3}} = \frac{16}{25}$. And $EF^2 = 30^2\left(\frac{16}{25}\right) = 24$.

S92J11 By the chord product theorem, (x)(1) = ($\sqrt{3}$)(4-$\sqrt{3}$)

x = 4$\sqrt{3}$ - 3

BD = x + 1 = 4$\sqrt{3}$ -2

S92J12 Comparing altitudes, $(x-1)^2 - y^2 = (x+1)^2 - (x-y)^2$. Solving, $y = \frac{x-4}{2}$.

The height = $\sqrt{(x-1)^2 - \left[\frac{x-4}{2}\right]^2} = \sqrt{\frac{3(x^2-4)}{2}}$. Thus, $\sqrt{3(x^2-4)} = 2n$ (n is an integer). Clearly x must be even and greater than 2. By trial and error, x = 4 and 14.

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CONTEST 3 - SOLUTIONS

S92J13 (a + bi)(c + di) = 1 = 1 + 0i = (ac – bd) + (ad +bc) i. $\frac{a}{b} = \frac{c}{d} \rightarrow$ ad = bc, and $b \neq 0 \neq d$. Equating real and imaginary parts, ac – bd = 1, ad + bc = 0. Since $b \neq 0 \neq d$, the latter leads to iad = 0 $\rightarrow$ a = 0. Hence, also c = 0. Thus, (b + ai)(c + di) = bdi. But (0)(0) – bd = 1 implies that b = -1. Thus, the answer is – i.

S92J14 Circumscribe a circle about $\triangle ABC$. $\angle C$ is an inscribed angle of the circle so that $m\overset{\frown}{AB} = 60^\circ$. Thus, $\angle ADB$ is a central angle (since its measure is 60 and AD = DB) so D is the center of the circle. Since $m\angle B = 70^\circ$, $m\overset{\frown}{AC} = 140^\circ$, so that $m\angle ADC = 140^\circ$.

S92J15 Factoring gives (x – 35)(x + 35). To be prime, one factor must be either +1 or –1. The smallest x giving a positive prime is therefore –6 (giving 41).

S92J16 Note that one minute is equal to six degrees so that five minutes is 30° . This means that each minute, the angle between the hands changes by 5.5° . Our time must therefore be between 1:00 and 2:00 since our angle has a measure of 180° or less. Calculating for reference:

Time	Angle
1:15	52.5°
1:30	135°
1:40	170°
1:45	142.5°

We see there are two such times, one between 1:15 and 1:30 and the other between 1:40 and 1:45.

115 + x = 52.5 + 5.5x $\rightarrow$ x = $13\frac{8}{9}$ minutes $\rightarrow 1{:}28{:}\frac{8}{9}$

140 + x = 170 – 5.5x $\rightarrow$ x = $4\frac{8}{13}$ minutes $\rightarrow 1{:}44{:}\frac{8}{13}$

S92J17 $\sqrt{\frac{n(n+1)}{2}} = m \rightarrow n(n+1) = 2m^2$.

Since n and n+1 are relatively prime, either $n = a^2$, $n+1 = a^2+1 = 2b^2$ or $n+1 = a^2$, $n = a^2 - 1 = 2b^2$. Make a table:

x^2	1	4	9	16	25
$2x^2$	2	8	18	32	50
	*	*			*
	=	=			=
	1+1	9-1			49+1

Thus, we have: 1, 8, 49.

S92J18 There are 3 equally likely classes:

<u>A opposite B</u>: The sum $mAC + mBD < 180^\circ$. If $m\angle BED < 90^\circ$, $\left[mBED = \dfrac{mAC + mBD}{2} \right]$. This has a probability ½ [interchange C and D].

<u>A opposite D</u>: Probability is ½ as above.

<u>A opposite C</u>: The sum of the angle measures is 360°, which is more than 180°. Thus, we ALWAYS have a sum bigger than 180° so that the probability is 1.

The answer is found by multiplying and adding: $\frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}(1) + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{2}{3}$.