NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 1

PART 1: 10 MINUTES

F91S1 Find all the real values of y satisfying $\left|\sqrt{y}-\sqrt{3}\right|<1$.

F91S2 Square ABMN is constructed on the hypotenuse of right triangle ABC. If the measure of AC is 1 and the measure of BC is 20, compute the measure of line segment MC.

PART 2: 10 MINUTES

F91S3 Let f be a function of x only and let h be a function of y only. Determine h such that log(f) + log(h) = log(1 + z) where z = x + xy + y.

F91S4 Suppose X and Y are three digit integers such that Y is the integer formed by reversing the digits of X and X >Y. If X - Y is divisible by 5 and the sum of the digits of X - Y is equal to the sum of the digits of X, find all possible values of X.

PART 3: 10 MINUTES

F91S5 If f(x) = 1990 - 5x, then how many pairs (a, b) of integers are there with $0 \le a \le 1000$, $0 \le b \le 1000$ and f(2a) = f(a + b) - f(a - b)?

F91S6 Define a sequence of positive numbers as $a_0 = 1/2$ and $a_n{}^2 = \frac{a_{n-1}+1}{2}$ for $n \ge 1$. a_{10} can be expressed as $\cos\beta$. What is the least positive β in degrees?

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 2

PART 1: 10 MINUTES

F91S7 Find the units digit of the sum 1! + 2! + … + 14! + 15!.

F91S8 In triangle ABC, BD is a median. CF intersects BD at E so that line segment BE equals line segment ED. Point F is on line AB. If line segment BF equals 5, find the measure of line segment BA.

PART 2: 10 MINUTES

F91S9 If $x^2 + y^2 = c^2$, find the minimum value of $x^4 + y^4$ in terms of c.

F91S10 Consider an intersection point of the polar graph of $r^2 = 8\sin 2\theta$ and the Cartesian graph of y = 1/x. How far is this point from the origin?

PART 3: 10 MINUTES

F91S11 How many four digit positive integers are divisible by 12 and have all of its digits prime?

F91S12 Find all integers n such that (x - n)(x - 17) + 4 = 0 has two integral roots.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 3

PART 1: 10 MINUTES

F91S13 For what positive integral base b does $21_b \cdot 54_b = 1354_b$?

F91S14 A circle of radius 1/2 is randomly dropped onto the Cartesian plane. Find the probability that point (i, j), where i, j are any integers.

PART 2: 10 MINUTES

F91S15 Compute the value of $(\cos 10^\circ)(\cos 20^\circ - 2\sin^2 10^\circ)$.

F91S16 Find the least positive integer z such that 546x + 1365y = 10^6 + z for some integers x and y.

PART 3: 10 MINUTES

F91S17 Find the sum of all the positive integers N less than 10 for which N! + (N+1)! Is divisible by 5.

F91S18 In triangle ABC, AB = 3, BC = 8, and AC = 7. Equilateral triangles ABP, BCQ, and CAR are exterior to triangle ABC. Compute the value of PC + QA + RB.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 4

PART 1: 10 MINUTES

F91S19 Find the sum of the digits of the integers between 1 and 1000 inclusive.

F91S20 A triangle has two medians of length 5 and 6 and an area pf 12. Find all possible lengths of the third median.

PART 2: 10 MINUTES

F91S21 Find the smallest positive integer whose cube ends in the digits 03.

F91S22 Find the integral value of n such that $x^2 - x + n$ is a factor of $x^9 + 342x - 6461$.

PART 3: 10 MINUTES

F91S23 Solve the following system for (a, b, c) if a > 0, b > 0, and c > 0.
$ab + bc + ca + a^2 = 182$
$ab + bc + ca + b^2 = 294$
$ab + bc + ca + c^2 = 273$.

F91S24 A regular n-gon is inscribed in a circle of radius r. Find all n such that the area of the n-gon is an integral multiple of r^2.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 5

PART 1: 10 MINUTES

F91S25 Suppose f(x, y) satisfies the following conditions:
(i) f(x, y) = f(x + y, y)
(ii) f(x, y) = f(y, x)
(iii) f(x, x) = x,
where x and y must be positive integers. Find f(89, 55).

F91S26 What is the probability of getting exactly three heads in 6 tosses of a coin if we know for sure that at least one head will appear in the first three tosses?

PART 2: 10 MINUTES

F91S27 Let $f(x) = x^2 + 1990x + 36$. How many pairs (a, b) are there such that a, b are positive integers and $f(a+b) = f(a) + f(b)$.

F91S28 Find all positive integers which have:
(i) exactly three distinct prime factors
(ii) exactly eight positive integral divisors
(iii) the sum of its positive integral divisors is equal to 3696.

PART 3: 10 MINUTES

F91S29 Find the least positive solution in degrees of $\tan\theta + \sec 2\theta = 1$.

F91S30 Find all points (x, y) in the first quadrant on the hyperbola xy = 16 such that there exists a real z satisfying $x^{x+y} = y^{9z}$ and $y^{x+y} = x^z$.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 1 - SOLUTIONS

F91S1 We know $-1 < \sqrt{y} - \sqrt{3} < 1$.
$\sqrt{3} - 1 < \sqrt{y} < \sqrt{3} + 1$
$4 - 2\sqrt{3} < y < 4 + 2\sqrt{3}$.

F91S2 Using the Law of Cosines, we get
$MC^2 = BC^2 + BM^2 - 2(BC)(BM)\cos\angle MBC$
$= BC^2 + (AC^2 + BC^2) - 2(BC)(BM)\cos(\angle ABC + 90^\circ)$
$= 2BC^2 + AC^2 + 2(BC)(BM)\sin\angle ABC$
$= 2BC^2 + AC^2 + 2(BC)(BM)\left(\dfrac{AC}{BM}\right)$
$= 2BC^2 + AC^2 + 2(BC)(AC)$
$= 2(400) + 1 + 2(20) = 841$. Therefore, $MC = 29$.

F91S3 $\log(1+z) = \log(1 + x + xy + y) = \log(1+x)(1+y)$
$= \log(1+x) + \log(1+y) = \log(fh)$. Therefore, $h(y) = 1 + y$.

F91S4 Let $X = abc_{10}$ and $Y = cba_{10}$ where a, b, c are digits with a and c are not equal to 0. Now $X - Y = 99(a-c)$, thus $5 \mid (a-c)$. If a = c, then the sum of the digits of X -Y is 0, thus a cannot equal c. Now a - c must equal 5, which implies X - Y = 495 and thus the sum of the digits of X is 18. Now (a, c) can be (9, 4), (8, 3) (7, 2) or (6, 1), thus the answers are 954, 873, and 792.

F91S5 We need -10a + 1990 = -5a - 5b + 1990 + 5a -5b - 1990.
1990 = 10a - 10b which yields a - b = 199. We seek the number of pairs in the list (199, 0), (200, 1), … (1000, 801) which is 802.

F91S6 If $a_n = \cos\beta$, then $a_{n+1} = \dfrac{1+\cos\beta}{2} = \cos\left(\dfrac{\beta}{2}\right)$. It can be seen by induction that if $a_0 = \cos A$, then $a_n = \cos\dfrac{A}{2^n}$. Since $a_0 = \cos 60^\circ$,
$a_{10} = \cos\left(\dfrac{60^\circ}{2^{10}}\right) = \cos\left(\dfrac{15}{256}\right)$.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 2 - SOLUTIONS

F91S7 Since 5!, 6!, 7!, 8!, …. 15! all have factors of 2 an 5 they contain at least one factor of 10. Therefore, they all end in 0. Thus, the units digit of 1! + 2! +…+14! + 15! Is the unit digit of 1 + 2 + 6 + 4 = 13. The units digit is therefore 3.

F91S8 Select point M on EC such that line segment FE equals line segment EM. Connect D with M. Figure FDMB is a parallelogram with DM = 5, AF =10, and BA = 15.

F91S9 Let $m = x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2$. Since $2x^2y^2 > 0$, m is minimum when $2x^2y^2$ is maximum. Since $x^2 + y^2 = c^2$, x^2y^2 is a maximum when $x^2 = y^2 = \frac{c^2}{2}$. The maximum value of $2x^2y^2 = 2\left(\frac{c^2}{2}\right)\left(\frac{c^2}{2}\right) = \frac{c^4}{2}$. Hence, $= (c^2)^2 - \frac{c^4}{2} = \frac{c^4}{2}$.

F91S10 Change $r^2 = 8\sin 2\theta$ to a Cartesian equation. $(x^2 + y^2)^2 = 16xy$. Since $xy = 1$, any intersection point satisfies $(x^2 + y^2)^2 = 16$. Take the fourth root of both sides to obtain $(x^2 + y^2)^{1/2} = 2$.

F91S11 If an integer is divisible by 12, then it is divisible by 3 and 4. For an integer to be divisible by 4 then its two most right digits (considered as a two digit number) must be divisible by 4. Thus, the last two digits are 32, 52, or 72. If a number is to be divisible by three then the sum of its digits must be divisible by three. Our prospects have one of the forms TH32, TH52, or TH72, where T and H are elements of {2, 3, 5, 7}. Thus, the answer is the number of members of the following list that are divisible by three as T ranges through its possible values: T + 7, T + 9, T + 11, T + 8, T + 10, T + 12, T + 10, T + 12, T + 14, T + 12, T + 14, and T + 16. Trial and error can derive the answer or notice that the list can be divided into three groups of four binomials each that are congruent to T, T + 1, and T + 2 modulo 3. Thus, 2, 3, 5, and 7 can be applied to exactly one group of four giving the answer to be 4 x 4 = 16.

F91S12 We have $x^2-(n+17)x+17n+4=0$. So $x=\frac{[(n+17)\pm((n-17)^2-16)^{1/2})]}{2}$.

We need $(n-17)^2-16=m^2$. For some integer m or equivalently $(n-17)^2-m^2=(n-17+m)(n-17-m)=16$. We need to divide 16 into 2 even factors. Possibilities are (8)(2), (-8)(-2), (4)(4), (-4)(-4). In these situations n = 22, 12, 21, and 13. By inspecting our expression for x it is clear that the parity of n does not matter.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 3 - SOLUTIONS

F91S13

$21_b \cdot 54_b = 1354_b$

$(2b+1)(5b+4) = b^3 + 3b^2 + 5b + 4$

$10b^2 + 13b + 4 = b^3 + 3b^2 + 5b + 4$

$7b^2 + 8b = b^3$

$b^3 - 7b^2 - 8b = 0$

$b(b-8)(b+1) = 0$

$b = 0$ or 8 or -1. Note that by observation the base must be at least 6 and less than 10 since (21)(54) = 1134 in base 10.

F91S14

By concentrating on the random location of the circle's center, it can be seen that this center must be within a distance of 1/2 from a lattice point. Thus, the probability can be viewed as the following shaded region to that of the unit square. Therefore, the answer is $\pi / 4$.

F91S15

$(\cos 10^\circ)(\cos 20^\circ - 2\sin^2 10^\circ) = (\cos 10^\circ)(\cos 20^\circ - 1 + 1 - 2\sin^2 10^\circ) =$

$= (\cos 10^\circ)(\cos 20^\circ - 1 + \cos 20^\circ) =$

$= 2\cos 10^\circ \cos 20^\circ - \cos 10^\circ$

Using the identity $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, we get

$2\cos 10^\circ \cos 20^\circ - \cos 10^\circ = \cos 30^\circ + \cos 10^\circ - \cos 10^\circ = \dfrac{\sqrt{3}}{2}$.

F91S16

We have $273(2x+5y) = 10^6 + z$. Since $2(-2a) + 5(a) = a$ it is obvious that $546x + 1365y$ takes on values which are multiples of 273. It is not hard to verify that $10^6 \equiv 1$ (mod 273) so the least multiple of 273 which is greater than 10^6 is $10^6 + 272$. Therefore, the answer is 272.

F91S17

$N! + (N+1)! = N! + N!(N+1) = N!(1+N+1) = N!(N+2)$

$N!(N+2)$ will be divisible by 5 when $N \geq 5$ or N + 2 = 5. Therefore, the values of N less than 10 for which $N! + (N+1)!$ is divisible by 5 are 3, 5, 6, 7, 8, and 9. The sum, 3 + 5 + 6 + 7 + 8 + 9 = 38.

F91S18

Note that PC = QA = RB. In the figure, imagine rotating 60° counterclockwise about point A. After this rotation R will be at C's original position and B will be at P's original position. Therefore, RB = PC. A similar argument shows that RB = PC = QA. Looking at cosines of angles it is found that $m\angle ABC = 60^\circ$, so $m\angle PBC = 120^\circ$ with PB = 3 and BC = 8. By the Law of Cosines $PC^2 = 3^2 + 8^2 - 2(3)(8)(-1/2) = 97$. Thus, the answer is $3(97)^{1/2}$.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 4 - SOLUTIONS

F91S19 Think of the list 000, 001, 002, …, 999. This list contains 1000 integers or 3000 digits. By symmetry, each digit occurs exactly 300 times. Thus, the sum of the digits of the integers in [1, 999] is 300(0 + 1 + 2 + ..+ 9) = 13500. Include 1000 and the sum is 13501.

F91S20 The medians of a triangle form a triangle which has 3/4 the area of the original triangle. If x is the third median then we know that there is a triangle with sides 5, 6, and x with area (3/4)(12) = 9. Let θ be the angle in this "median triangle" between the sides of the length 5 and 6. Then $\frac{1}{2}\cdot 5\cdot 6\cdot \sin\theta = 9$ or $\sin\theta = 3/5$ implying that $\cos\theta = \pm 4/5$. Using the Law of Cosines, we get $x^2 = 5^2 + 6^2 \pm 2(5)(6)(4/5)$ or $x = \sqrt{13}, \sqrt{109}$.

F91S21 There is no one digit number whose cube ends in 03. Let $n = 10x + y$. Now, $n^3 = 1000x^3 + 300x^2y + 30xy^2 + y^3$. Since $n^3 \equiv 30xy^2 + y^3 (\text{mod}100) \equiv 30(7^2)x + 7^3 (\text{mod}100) \equiv 70x + 43 (\text{mod}100)$. We wish for $7x + 4$ to have a unit digit of 0. Obviously, $x = 8$. So n = 87. In fact $87^3 = 658503$.

F91S22 If $f(x)$ is a factor of $g(x)$ then for any integer $m, f(m)$ is a factor of $g(m)$ or $|g(m)|$. Let $x = 0$ and 1 to get $n | 6461$ and $n | 6118$. Now if n divides both these integers then n divides their differences which is $343 = 7^3$. Since 49 does not divide either integer we know n = -1, 1, -7, 7. One way to proceed is let x = -2, then $(n+6) | 7657$ and ± 7 are the only choices. If $x = -1$, then $(n+2) | 6804$ and $n = -7$ is impossible. Thus, $n = 7$.

F91S23 We actually have the system

$$(a+b)(a+c) = 2\cdot 7\cdot 13$$
$$(a+b)(b+c) = 2\cdot 3\cdot 7^2$$
$$(a+c)(b+c) = 3\cdot 7\cdot 13$$

Multiplying, we get $[(a+b)(b+c)(c+a)]^2 = 2^2\cdot 3^2\cdot 7^4\cdot 13^2$.
$(a+b)(b+c)(c+a) = 2\cdot 3\cdot 7^2\cdot 13$. (Note: $(a+b)(b+c)(c+a) > 0$)
Now divide the last equation by all three previous equations to get $b + c = 21$, $c + a = 13$, and $a + b = 14$. Solving this system we obtain the triple $(3,11,10)$.

F91S24 Divide the n-gon into n triangles with sides of length r and r with an included angle of $\frac{360^\circ}{n}$. The area of one of these triangles is $\frac{r^2}{2}\sin\frac{360^\circ}{n}$. Thus, the area of the n-gon is $\frac{nr^2}{2}\sin\frac{360^\circ}{n}$. We wish for $\frac{n}{2}\sin\frac{360^\circ}{n}$ to be an integer. For $n \in \{3,4,....12\}$, only a 4-gon and 12-gon satisfy the condition. The expression represents the area of a n-gon in a unit circle so $\frac{n}{2}\sin\frac{360^\circ}{n} < \pi$ because a unit circle has area π. Now, the 12-gon has area 3 in the unit circle, so for n > 12 we know that $3 < \frac{n}{2}\sin\frac{360^\circ}{n} < \pi$. Thus, the answer is 4 and 12.

NYCIML
SENIOR A DIVISION
FALL 1991

CONTEST 5 - SOLUTIONS

F91S25 Apply conditions (i) and (ii) repeatedly until you can apply condition (iii). This yields: f(89, 55) = f(34, 55) = f(55, 34) = f(21, 34) = f(34, 21) = f(13, 21) = f(21, 13) = f(8, 13) = f(13, 8) = f(5, 8)= f(8, 5) = f(3, 5) = f(5, 3) = f(2, 3) = f(3, 2) = f(1, 2) = f(2, 1) = f(1, 1) = 1. Actually, f is uniquely determined to be f(x, y) = gcd(x, y) and thus, f(89, 55) = 1.

F91S26 There are seven possibilities for the first three tosses -- three which have one head, three which have two heads, and one which has three heads. There are eight possibilities for the last three tosses. Thus, the sample space has 56 outcomes which are all equally likely. Now, if one head appears in the first three tosses, we need two heads in the last three tosses -- a total of (3)(3) = 9 possible ways. If two heads appear in the first three tosses, then we need one head later -- (3)(3) = 9 possible ways. If 3 heads appear in the first three tosses, we need 0 heads in the last three -- (1)(1) = 1 way. Thus, the answer is (9 + 9 +1)/56 = 19/56.

F91S27 We need $(a+b)^2+1990(a+b)+36=a^2+1990a+36+b^2+1990b+36$. We have $2ab=36$ or $ab=18$. Now, $18=(3^2)(2)$ has $(2+1)(1+1)=6$ factors, thus (1, 18), (2, 9), (3, 6), (6, 3), (9, 2), and (18, 1) are the only pairs. Therefore, the are six pairs.

F91S28 The fact that a number has 3 prime factors and 8 divisors means that the sought numbers have no repeated prime factors. If a number's prime factorization is $p_1p_2p_3$, then the sum of its positive divisors is $(p_1+1)(p_2+1)(p_3+1)$. We wish to find primes p_1, p_2, p_3 such that $(p_1+1)(p_2+1)(p_3+1)=(2)(2)(2)(2)(7)(11)$. Trying the various situations we see that either 5, 13, 43 or 2, 3, 307. Thus, the only answers are 2795 and 1842.

F91S29 We have $\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos 2\theta}=1$.

$\sin\theta\cos 2\theta+\cos\theta=\cos\theta\cos 2\theta$
$\sin\theta\cos 2\theta=(\cos\theta)(\cos 2\theta-1)$
$\sin\theta\cos 2\theta=(\cos\theta)(-2\sin^2\theta)$

Now, either $\sin\theta=0$ or $\cos 2\theta=-2\sin\theta\cos\theta=-\sin 2\theta$. Therefore, $\tan 2\theta=-1$ and $2\theta=135^\circ$ or $\theta=67.5^\circ$.

F91S30 Since x and y are positive, we can take the logs of the two equations, using base x and y, respectively. We obtain $x+y=9z\log_x y$ and $x+y=z\log_y x$. Now, multiply these new equations remembering that $(\log_y x)(\log_x y)=1$ to arrive at $(x+y)^2=9z^2$. Therefore, $x+y=\pm 3z$ and $\log_y x=\pm 3$. Now $x=y^3$ or $x=1/y^3$. Since $xy=16$, we know that $y^4=16$ or $y^{-2}=16$ and y is positive, y = 2 or y =1/4. The answer is therefore, (8, 2) and (64, 1/4).