NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION   CONTEST NUMBER ONE   SPRING, 1991

Part I: 10 Minutes

S91B1. John drew a card from a standard deck of 52 cards, and without replacing it, drew another card. What is the probability he drew a pair?

S91B2. A 3-digit number has the following properties. If you add 14 to it, the resulting number is divisible by 14. If you add 8 to it, the resulting number is divisible by 8. If you add 18 to it, the resulting number is divisible by 18. What is the number?

Part II: 10 Minutes   NYCIML CONTEST ONE   SPRING, 1991

S91B3. Find the sum of the first 100 odd integers.

S91B4. Working alone, a painter can paint a room in 10 hours. His helper can paint the same room in 15 hours. How long would it take them if they worked together?

Part III: 10 Minutes   NYCIML CONTEST ONE   SPRING, 1991

S91B5. Compute the sum of the distances from one vertex of a square to the midpoints of all 4 sides, if a side has length one.

S91B6. If A cows give B cans of milk in C hours, in terms of A, B, C, D, and E, how many hours will it take D cows to give E cans of milk?

ANSWERS

1. 1/17
2. 504
3. 10,000
4. 6 hours
5. $1 + \sqrt{5}$
6. $\frac{ACE}{BD}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER TWO SPRING, 1991

Part I: 10 Minutes

S91B7. A man earns \$259 in a 7-day week, each day earning \$5 more than he had earned on the previous day. How much did he earn on the first day?

S91B8. If $\sin\theta = \frac{1}{4}$ and θ is acute, compute the value of $\sin 2\theta$.

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1991

S91B9. Express 1001101_{two} as an equivalent numeral in base 3.

S91B10. David leaves the house at noon, travelling east at 40 mph. At 2:30 P.M., his wife leaves, taking the same route at 60 mph. At what time will she overtake him?

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1991

S91B11. If $f(x-1) = x^2 + 3x + 2$, find $f(x+1)$.

S91B12. If three fair dice are thrown, find the probability that the sum of the number of pips on the tops of the dice is exactly 6.

ANSWERS

7. \$22

8. $\frac{\sqrt{15}}{8}$

9. 2212 or 2212_{three}

10. 7:30 P.M.

11. $x^2 + 7x + 12$

12. $\frac{5}{108}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER THREE SPRING, 1991

Part I: 10 Minutes

S91B13. Find the length of a chord which is the perpendicular bisector of a radius, if the circle has a diameter of 20.

S91B14. If P is a 2-digit prime number, find the largest number which divides all numbers of the form P^2-1.

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1991

S91B15. John has 24 ounces of a solution which is 10% acid. How many ounces of pure acid should he add to make it a 40% solution?

S91B16. Compute the remainder when $x^{10} + x^5 + x$ is divided by $x + 1$.

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1991

S91B17. On Monday, a stock went up 10% in value. On Tuesday, it dropped 10% of its new value. On Wednesday, it rose 10%. On Thursday, it dropped 10%. On Friday, it rose 10%. What was the exact percentage gain for the week?

S91B18. If $i^2 = -1$, compute the value of $(1 + i)^{16}$.

ANSWERS

13.	$10\sqrt{3}$ or equivalent	15.	12	17.	7.811%
14.	24	16.	−1	18.	256

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FOUR SPRING, 1991

Part I: 10 Minutes

S91B19. If $P_1 + P_2 = 126$, where P_1 and P_2 are prime numbers, find the largest possible value for $P_1 - P_2$.

S91B20. Express $$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}}$$ as a fraction $\frac{a}{b}$ in lowest terms.

Part II: 10 Minutes NYCIML CONTEST FOUR SPRING, 1991

S91B21. In right triangle ABC, $\angle C = 90^\circ$, CM is a median. If $\angle B = 40^\circ$, find $\angle CMB$.

S91B22. Find the length of the edge of a cube if its volume is numerically equal to its surface area.

Part III: 10 Minutes NYCIML CONTEST FOUR SPRING, 1991

S91B23. If $K = \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{d}{|d|} + \frac{abcd}{|abcd|}$,

a, b, c, d $\neq$ 0, and real, find all possible values of K.

S91B24. If $S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{19}$, find S.

ANSWERS

19. 100
20. $\frac{169}{70}$ or equivalent
21. 100 or 100°
22. 6
23. −3, 1, 5
24. 1,048,575

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE SPRING, 1991

Part I: 10 Minutes

S91B25. In triangle ABC, D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, G is the midpoint of $\overline{BD}$. If the area of $\triangle$ ABC is 24, find the area of $\triangle$ DGF.

S91B26. Find all ordered pairs of integers (x, y) such that $x + y = xy$.

Part II: 10 Minutes NYCIML CONTEST FIVE SPRING, 1991

S91B27. Find the number of sides of a regular polygon each of whose interior angles measures 170°.

S91B28. John is dealt 3 cards from a standard deck of 52 cards. What is the probability that they are all of the same suit?

Part III: 10 Minutes NYCIML CONTEST FIVE SPRING, 1991

S91B29. Find the digit K if the number 725,382,9K4,180 is divisible by 11.

S91B30. Cynthia invests \$K in a bond yielding 6 1/2 %, and \$(K + 1000) in a bond yielding 9%. If her income from the 9% bond is \$102 less than twice the income from the 6 1/2 % bond, find K.

ANSWERS

25.	1.5 or $\frac{3}{2}$ or $1\frac{1}{2}$	27.	36	29.	0
26.	(0,0) (2,2)	28.	$\frac{22}{425}$ or equivalent	30.	\$4800

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER ONE SPRING, 1991

SOLUTIONS

S91B1. Once he has drawn the first card, there are 51 remaining, of which 3 will produce the pair:

$$\frac{3}{51} = \frac{1}{17}$$

S91B2. The sentences say that the number is divisible by 14, 8, 18. The only 3 digit number that qualifies is $7 \times 8 \times 9 = 504$.

S91B3. Using the arithmetic progression formula,

$$S = \frac{100}{2}(1+199) = 10{,}000$$

or

The sum of the first N odd integers is N^2.

S91B4. In x hours, the painter completes x/10 of the room, the helper completes x/15 of the room.

$$\frac{x}{10} + \frac{x}{15} = 1 \qquad x = 6$$

S91B5. The distances to the midpoints of the adjacent sides are each 1/2. The distances to the midpoints of the new adjacent sides are each $\sqrt{5}/2$.

S91B6. The number of cans per cow-hour is a constant, that is,

$$\frac{B}{AC} = \frac{E}{DX} \qquad x = \frac{ACE}{BD}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER TWO SPRING, 1991

SOLUTIONS

S91B7. Using the formula for the sum of an arithmetic progression with x = the first term:

$$259 = \frac{7}{2}(2x + (7-1)\ 5)$$

$$518 = 14x + 210$$

$$22 = x$$

S91B8. If $\sin\theta = 1/4$, $\cos\theta = \sqrt{15}\,/\,4$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{1}{4}\cdot\frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{8}$$

S91B9. The easiest way is to change to base 10 and then to base 3.

$$1001101_2 = 77_{10} = 2212_3$$

S91B10. If x = number of hours the wife travelled,

$$40\left(x + \frac{5}{2}\right) = 60x$$

$$x = 5$$

Five hours from 2:30 is 7:30.

S91B11. Let $a = x-1$ so $x = a+1$

$$f(a) = (a+1)^2 + 3(a+1) + 2 = a^2 + 5a + 6$$

$$f(x+1) = (x+1)^2 + 5(x+1) + 6 = x^2 + 7x + 12$$

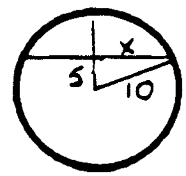
S91B12. There are $6^3 = 216$ possible outcomes. Of these, 10 have a sum of 6: 6 (or 3!) ways they can land (1, 2, 3), 3 ways (4, 1, 1), (1, 4, 1), (1, 1, 4) and (2, 2, 2).

$$\frac{10}{216} = \frac{\ }{3}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER THREE SPRING, 1991

SOLUTIONS

S91B13. $x = 5\sqrt{3}$



The chord is twice that, or $10\sqrt{3}$.

S91B14. $P^2-1 = (P+1)\cdot(P-1)$. If P is a 2 digit prime, P+1 and P-1 must be even, and one must be divisible by 4. Also, since P is not divisible by 3, either P+1 or P-1 must be. Since 2, 3, 4 are all factors, 24 must divide P^2-1.

S91B15. If x is the number of ounces of pure acid to be added, the number of ounces of acid in the final mix will be equal to the original plus what is added:

$.1(24) + x = .4(24 + x)$ and $x = 12$.

S91B16. Using the remainder theorem, the remainder is $f(-1)$:

$f(-1) = (-1)^{10} + (-1)^5 - 1 = -1$

S91B17. Using a model $1,000 stock:

1000 + 100 = 1100
1100 - 110 = 990
990 + 99 = 1089
1089 - 108.90 = 980.10
980.10 + 98.01 = 1078.11 This represents a gain of 7.811%.

S91B18. $(1 + i)^2 = 1 + 2i - 1 = 2i$

$(1 + i)^4 = (2i)^2 = 4i^2 = -4$

$(1 + i)^{16} = \left[(1 + i)^4\right]^4 = (-4)^4 = 256$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION   CONTEST NUMBER FOUR   SPRING, 1991

SOLUTIONS

S91B19 Obviously, P_1 and P_2 must be odd. Using pairs of numbers whose sum is 126, and working down from 123 - 3, 121 - 5, etc., the largest possible value where P_1 and P_2 are primes is 113 - 13 = 100.

S91B20 Working backwards from the bottom, $\frac{1}{2+\frac{1}{2}} = \frac{2}{5}$

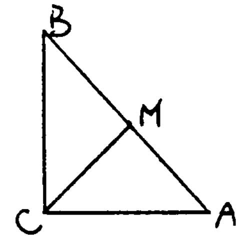
$\frac{1}{2+\frac{2}{5}} = \frac{5}{12}$ $\frac{1}{2+\frac{5}{12}} = \frac{12}{29}$ $\frac{1}{2+\frac{12}{29}} = \frac{29}{70}$

$2 + \frac{29}{70} = \frac{169}{70}$

S91B21 A median to the hypotenuse is 1/2 the hypotenuse. Therefore CM = MB.

B = 40° BCM = 40° and

CMB = 100°.



S91B22 $e^3 = 6e^2$ $e = 6$

S91B23 There are 5 possibilities.

If a, b, c, d are all positive, k=5.
If one is negative, k=1.
If 2 are negative, k=1.
If 3 are negative, k=-3.
If all are negative, k=-3.

S91B24 The sum of $1 + 2 + 2^2 + 2^3 + \quad 2^N = 2^{N+1} - 1$

$S = 2^{20} - 1 = 1{,}048{,}575$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE SPRING, 1991

SOLUTIONS

S91B25. Since a median divides a triangle into 2 triangles with equal area,

$$K\triangle DGF = \frac{1}{2}K\triangle DBF = \frac{1}{2}\left(\frac{1}{2}K\triangle BED\right) = \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{2}K\triangle ABD\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}K\triangle ABC = \frac{1}{16}K\triangle ABC$$

$K\triangle ABC = 24$ so $K\triangle DGF = 1.5$

S91B26. $xy = x + y$
$xy - x - y = 0$
completing the product of 2 binomials,
$xy - x - y + 1 = 1$
$(x - 1)(y - 1) = 1$
Either both factors are 1 or both are -1.
(0,0) (2,2)

S91B27. The exterior angle is 10. This is measured by

$$\frac{360}{N} = 10, \quad N = 36$$

OR

$$\frac{(N-2)\cdot 180}{N} = 170 \qquad 170N = 180N - 360$$

$$-10N = -360$$
$$N = 36$$

S91B28. The first could be any card. The second has a 12/51 probability of being the same suit. The third a probability of

$$\frac{11}{50}\cdot\frac{12}{51}\cdot\frac{11}{50} = \frac{22}{425}$$

OR

The number of three card possibilities is ${}_{52}C_3 = 22,100$
The number of successes is $4\cdot{}_{13}C_3 = 1144$.

$$\frac{1144}{22100} = \frac{22}{425}$$

S91B29. The sum of the odd positioned integers must differ from the sum of the even positioned integers by a multiple of 11.

$7 + 5 + 8 + 9 + 4 + 8 = 41$
$2 + 3 + 2 + K + 1 = K + 8$

$41 - (K + 8)$ or $33 - K$ must be a multiple of 11.
K must equal 0.

S91B30. Income = Principal X Rate
$.09(K + 1000) = 2(.065K) - 102$ Multiply by 100
$9K + 9000 = 13K - 10200$
$19200 = 4K$
$4800 = K$

May 7, 1991

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1991 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S91SA7	1 or 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$
	S91SA19	4825 or $2^{12}+3^{6}$
	S91SA22	4/3 and -3 or all real numbers
Senior B	S91B24	1,048,575 or $2^{20}-1$

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML