New York City Interscholastic Mathematics League

SENIOR DIVISION — CONTEST NUMBER ONE — SPRING 1991

Part I: 10 Minutes

S91SA1. Find the positive integer n for which $\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}} = n$

S91SA2. Find all values of cos x satisfying the equation

$$25 \cos 3x = 6 \cos x.$$

Part II: 10 Minutes

S91SA3. A circle of radius r is cut in half and each half is then folded into the shape of a cone. Compute the volume of each cone.

S91SA4. Solve for n:

$$\frac{\binom{2n}{n}}{n+1} = \frac{\binom{2n+2}{n+1}}{1990}$$

Part III: 10 Minutes

S91SA5. Compute the value of $x^3 + \frac{1}{x^3}$ if $x + \frac{1}{x} = \sqrt{3}$.

S91SA6. If brackets denote the greatest integer function, compute all real values of x such that

$$6x = 7 + \left[x^2 \right]$$

Answers

1. $n = 4$
2. $\cos x = +0.9, 0, -0.9.$
3. $\frac{\pi \cdot \sqrt{3}\, r^3}{24}$
4. $n = 497$
5. 0
6. 4/3, 3/2, 9/2 and 14/3, or equivalents [Must have all 4 answers.]

New York City Interscholastic Mathematics League

SENIOR DIVISION CONTEST NUMBER TWO SPRING 1991

Part I: 10 Minutes

S91SA7. Find all values of x satisfying the equation $x^{3/5} + 7x^{3/10} = 8$.

S91SA8. What is the positive value of tan x if $61 \tan x = 60 \sec x$?

Part II: 10 Minutes

S91SA9. If $x + x^{-1} = 2\sqrt{2}$, compute $[x^5 + x^{-5}]$ where the brackets signify the greatest integer function.

S91SA10. An equiangular dodecagon (12 sides) is inscribed in a circle. If two consecutive sides of this dodecagon have lengths $\sqrt{7}$ and $2\sqrt{21}$, compute the area of the circle.

Part III: 10 Minutes

S91SA11. Find the value of $\displaystyle\sum_{n=1}^{100} n(n!)$ [Answer may be left in simple factorial form.]

S91SA12. A cube whose edge has length n is assembled by fastening n^3 unit cubes together. The large cube is painted and then separated into the unit cubes again. Allow N(x) to represent the number of unit cubes which have exactly x faces painted. For how many values of n is the quantity $\dfrac{N(0)\,N(2)}{N(1)\,N(3)}$ a positive integer less than 1989 ?

Answers

7. 1, 1024
8. tan x = +60/11 or equivalent
9. 82
10. 133π
11. $101! - 1$
12. 44

New York City Interscholastic Mathematics League

SENIOR DIVISION — CONTEST NUMBER THREE — SPRING 1991

Part I: 10 Minutes

S91SA13. Assuming a limit, find the value of $\sqrt{3\sqrt{24\sqrt{3\sqrt{24\sqrt{3\ldots}}}}}$

S91SA14. Solve for all real values of tan x such that $3\tan x + 4 = \sec x$

Part II: 10 Minutes

S91SA15. If $42a^2 - 71ab + 30b^2 = 0$ find the POSITIVE integers a and b which satisfy the equation so that a+b is a minimum. Express your answer as (a,b).

S91SA16. If the geometric mean of $\log_2 17$, $\log_8 17$ and $\log_{512} 17$ is $\log_k 17$, then compute the value of k.

Part III: 10 Minutes

S91SA17. Let $f(x) = [x] + [2x] + [4x]$. If the solution set of $f(x) = 10$ is $\{x \text{ such that } a \le x < b\}$, find the ordered pair (a,b).

SA91SA18. Find n such that $1\cdot2 + 4\cdot5 + 7\cdot8 + \ldots + (3n-2)(3n-1) = 1199n$.

Answers

13. 6

14. $\frac{-6 \pm \sqrt{6}}{4}$

15. (5,6)

16. k = 8

17. $\left(\frac{3}{2}, \frac{7}{4}\right)$ or equivallent.

18. 20

New York City Interscholastic Mathematics League

SENIOR DIVISION CONTEST NUMBER FOUR SPRING 1991

Part I: 10 Minutes

S91SA19. Compute the cube root of $16^9 + 27^6 + 2^{17} \cdot 6^7 + 3 \cdot 6^{12}$.

S91SA20 Find all possible values of tan x such that $9 \sin x + 2 \cos x = 6$.

Part II: 10 Minutes

S91SA21. Assuming a limit, compute the value of $\sqrt{3\sqrt[3]{288\sqrt{3\sqrt[3]{288\sqrt{3\ldots}}}}}$

S91SA22. Find both possible values of $\log_y x$ if $x^4 - \left(\frac{x}{y}\right)^3 - xy^4 + y = 0$.

Part III: !0 Minutes

S91SA23. How many positive integers less than 100 can be expressed as [2x] + [3x] for some real number x.

S91SA24. A coin is flipped until a head appears. Then the coin is flipped again until a head appears once again. What is the probability that the number of flips ir the second experiment is at least three times the number of flips in the first?

Answers

19. 4825
20. 8/15 and −4/3
21. 6
22. −3 and 4/3
23. 79
24. 2/15

New York City Interscholastic Mathematics League

SENIOR DIVISION — CONTEST NUMBER FIVE — SPRING 1991

Part I: 10 Minutes

S91SA25. Assuming a limit, compute the value of x if

$$2^{x} = \sqrt[2]{2\sqrt[3]{2\sqrt[4]{2\sqrt[2]{2\sqrt[3]{2\sqrt[4]{2\sqrt[2]{2\ldots}}}}}}}$$

S91SA26. Suppose z is a root (possibly complex) of $z^4 + z^3 + z^2 + z + 1 = 0$. Compute the value of z^5.

Part II: 10 Minutes

S91SA27 Find all ordered pairs of positive integers (x,y)

such that $(3^{x})(8^{y}) = 2^{7x+2y+1} + 2^{2x+3y-1}$

S91SA28. Suppose that **f** is a function defined for positive values of x, such that

$$11f(x+1) + 5f(x^{-1}+1) = \log_{10} x.$$

Compute the value of f(6) + f(17) + f(126).

Part III: 10 Minutes

S91SA29. If [a] denotes the greatest integer less than or equal to a, find the least x such that

$$[x] + [5x] + [25x] + [125x] = 10^{3}.$$

S91SA30. Compute the probability of getting exactly 3 heads in 6 tosses of a coin if we know for sure that at least one head will appear in the first three tosses.

Answers

25. x = 17/23
26. 1
27. (1,8) and (2,15)
28. 2/3
29. $6\frac{52}{125}$ or equivalent
30. 19/56

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER ONE — SPRING 1991

SOLUTIONS

S91SA1. For simplicity, let $a = 26 + 15\sqrt{3}$ and $b = 26 - 15\sqrt{3}$. We wish to find n such that $a^{1/3} + b^{1/3} = n$. Cubing both sides and rearranging gives $a + b + 3a^{1/3}b^{1/3}(a^{1/3} + b^{1/3}) = n^3$. Substituting that $a+b = 52$, that $a^{1/3}b^{1/3} = (ab)^{1/3} = (26^2 - 3(15^2))^{1/3} = 1$ and that $a^{1/3} + b^{1/3} = n$, reduces the above equation to $n^3 - 3n - 52 = 0$. The only rational possibilities for n are 1, 2, 4, 13, 26, 52, −1, −2, −4, −13, −26, and −52. The only real answer is n=4.

S91SA2. The aim is to express cos 3x in terms of cos x:

$$\cos 3x = \cos(x+2x) = \cos x \cos 2x - \sin x \sin 2x$$
$$= \cos x\,(2\cos^2 x - 1) - \sin x\,(2\sin x \cos x)$$
$$= \cos x\,(2\cos^2 x - 1) - 2(1-\cos^2 x)\cos x$$
$$= \cos x\,(2\cos^2 x - 1 - 2 + 2\cos^2 x).$$ This gives

$$\boxed{\cos 3x = \cos x\,(4\cos^2 x - 3)}$$ (This is a famous identity)

The original equation becomes $25\cos x\,(4\cos^2 x - 3) = 6\cos x$

<u>Case 1:</u> $\cos x = 0$ which is obviously a possibility.

<u>Case 2:</u> $\cos x \neq 0$: Divide both sides by cos x to get $4\cos^2 x - 3 = 6/25$

This reduces to $\cos^2 x = 81/100$ so $\cos x = +0.9$ or -0.9.

S91SA3. The circumference of the circular base of each cone is πr. Thus, each cone has radius r/2 and slant height r. By the Pytagorean Theorem, $h^2 + (r/2)^2 = r^2$ where h is the length of the altitude of the cone. This yields $h = \frac{r}{2}\sqrt{3}$ so that the volume of the cone is $\frac{\pi}{3}(r/2)^2 h$.

This simplifies to $\frac{\pi}{3}(r^2/4)\left(\frac{r}{2}\sqrt{3}\right) = \frac{\pi}{24}\sqrt{3}\,r^3$

S91SA4. The given equation is equivalent to

$$1990\,\frac{(2n)!}{n!n!} = (n+1)\,\frac{(2n+2)!}{(n+1)!(n+1)!}$$

This gives $1990(n+1) = (2n+2)(2n+1) \longrightarrow 1990 = 2(2n+1)$ Thus $n = 497$.

S91SA5. $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$ Thus we have

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = (\sqrt{3})^3 - 3(\sqrt{3}) = 0.$$

S91SA6. $[x^2] = 6x - 7$ implies that $x = \frac{a}{6}$ for some set of integers a.

Substitute to get $[a^2/36] = a - 7$. This implies that $a - 7 \le \frac{a^2}{36} < a - 6$ by the definition of the greatest integer function. Solving for the integers which satisfy the inequality yields a can be any member of the set {8, 9, 27, 28} so that the solution set for x is {4/3, 3/2, 9/2, 14/3}.

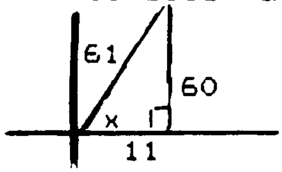
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER TWO SPRING 1991

SOLUTIONS

S91SA7. Factoring, we get $(x^{3/10} + 8)(x^{3/10} - 1) = 0$
This gives $x^{3/10} = -8$ or $x^{3/10} = 1$ so $x = 1024, 1$

S91SA8. We have $61 \dfrac{\sin x}{\cos x} = 60 \dfrac{1}{\cos x}$. Since cos x is not zero, we can multiply both sides by cos x to get $61 \sin x = 60$, or $\sin x = 60/61$
Since we are looking for the positive value of tan x, we can consider Quadrant I only and this reference diagram:

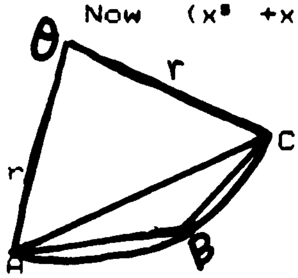


Thus $\tan x = \dfrac{+60}{11}$

S91SA9. $x^5 + x^{-5} = (x^3 + x^{-3})(x^2 + x^{-2}) - (x + x^{-1}) =$
$[(x^2 + x^{-2})(x + x^{-1}) - (x + x^{-1})](x^2 + x^{-2}) - (x + x^{-1})$
Note that $(x^2 + x^{-2}) = (x + x^{-1})^2 - 2 = 6$

Thus $x^5 + x^{-5} = (12\sqrt{2} - 2\sqrt{2})(6) - 2\sqrt{2} = 58\sqrt{2}$.
Now $(x^5 + x^{-5})^2 = 6728$, so $[x^5 + x^{-5}] = 82$

S91SA10.



Let $AB = \sqrt{7}$ and $BC = 2\sqrt{21}$. Since the dodecagon is equiangular, $m\angle ABC = 150°$. The law of cosines (in $\triangle ABC$ gives $AC^2 = 133$. Since the polygon is equiangular, every two consecutive sides are included in an arc of sixty degrees so that $m\angle AOC$ is 60. Thus $\triangle AOC$ is equiangular so that $AC^2 = r^2 = 133$ so that the area is 133π.

S91SA11.
NOTE: $$\sum_{n=1}^{100} (n+1)n! = \sum_{n=1}^{100} (n+1)! = \sum_{n=2}^{101} n!$$

ALSO: $$\sum_{n=1}^{100} (n+1)n! = \sum_{n=1}^{100} n(n)! + \sum_{n=1}^{100} n!$$

Set these equal and solve for $\sum_{n=1}^{100} n(n!)$

Thus $$\sum_{n=1}^{100} n(n!) = \sum_{n=2}^{101} (n)! - \sum_{n=1}^{100} n! = 101! - 1$$

S91SA12. First realize that $N(0) = (n-2)^3$, $N(1) = 6(n-2)^2$, $N(2) = 12(n-2)$ and $N(3) = 8$. So the needed quantity has a value of $\dfrac{(n-2)^2}{4}$. Solve $\dfrac{(n-2)^2}{4} < 1989$, where n is a positive integer. Thus we get $n \le 91$. Note that for $(n-2)^2/4$ to be an integer, n must be even. Also, the quantity has no value if n=2, so that the answers sought are 4, 6, 8, ..., 90. Thus there are 44 values. [Note that n MUST be integral since there are n^3 cubes.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER THREE SPRING 1991

SOLUTIONS

S91SA13.

Let $x = \sqrt{3\sqrt{24\sqrt{3\sqrt{24\sqrt{3 \ldots}}}}}$ Then $x = \sqrt{3\sqrt{24x}}$ which means that $x^2 = 3\sqrt{24x}$ and $x^4 = 216x$. Thus $x(x^3 - 216) = 0$ Since x is not zero, x must be 6.

S91SA14. If $3\tan x + 4 = \sec x$ we can square both sides to obtain $9\tan^2 x + 24\tan x + 16 = \sec^2 x$ Use the fact that $\sec^2 x = \tan^2 x + 1$ and substitute to get $9\tan^2 x + 24\tan x + 16 = \tan^2 x + 1$. Thus $8\tan^2 x + 24\tan x + 15 = 0$. Using the quadratic formula, we get

$$\tan x = \frac{-6 \pm \sqrt{6}}{4}$$

S91SA15. If $42a^2 - 71ab + 30b^2 = 0$, then factoring gives $(6a-5b)(7a-6b) = 0$. Thus $6a = 5b$ or $7a = 6b$. Since the greatest common divisor of 6 and 5 is 1 and the greatest common divisor of 7 and 6 is also 1, positive integral solutions for (a, b) are of the form (5k, 6k) or (6k, 7k) with k = 1, 2, 3, ... Thus for a+b to be a minimum take k to be 1 giving (5, 6).

S91SA16. From the given and by the change of base theorem we have:
$\log_k 17 = (\log_2 17 \log_8 17 \log_{512} 17)^{1/3} = \left(\frac{\log 17}{\log 2}\frac{\log 17}{\log 8}\frac{\log 17}{\log 512}\right)^{1/3}$

$$= \left(\frac{\log^3 17}{(\log 2)(3\log 2)(9\log 2)}\right)^{1/3} = \frac{\log 17}{3\log 2} = \frac{\log 17}{\log 8} = \log_8 17$$

which means that $k = 8$

S91SA17. Note that for any integer n, <u>any</u> number in the interval $\left[\frac{n}{4}, \frac{n+1}{4}\right)$ has the same image under f(x). Another way to look at this is that the changes in the values of f(x) occur exactly at the numbers n/4, where n is an integer. Obviously, if f(x) = 10, then x is between 1 and 2. Note that f(1) = 7, f(5/4) = 8, f(3/2) = 10, f(7/4) = 11. Thus the answer is (3/2, 7/4).

S91SA18. We wish to solve $\sum_{k=1}^{n}(3k-2)(3k-1) = 1199n$.

$$\sum_{k=1}^{n}(3k-2)(3k-1) = 9\sum_{k=1}^{n}k^2 - 9\sum_{k=1}^{n}k + 2\sum_{k=1}^{n}1 =$$

$$9\left(\frac{n(n+1)(2n+1)}{6}\right) - 9\left(\frac{n(n+1)}{2}\right) + 2n = n(3n^2 - 1).$$

Now, $n(3n^2 - 1) = 1199n$ has three solutions: −20, 0, and 20. Only 20 is acceptable.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER FOUR SPRING 1991

SOLUTIONS

S91SA19. Rewrite the problem as $2^{36} + 3^{18} + 3(2^{24})(3^{6}) + 3(2^{12})(3^{12})$

$= (2^{12})^3 + 3(2^{12})^2(3^6) + 3(2^{12})(3^6)^2 + (3^6)^3$

Since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, the answer is $2^{12} + 3^6 = 4825$.

S91SA20. Assume that cos x is not zero and divide both sides by cos x giving $9\tan x + 2 = 6\sec x$. Squaring both sides gives $81\tan^2 x + 36\tan x + 4 = 36\sec^2 x$. Now use $\sec^2 x = \tan^2 x + 1$ to arrive at the quadratic $45\tan^2 x + 36\tan x - 32 = 0$ with solutions 8/15 and −4/3.

S91SA21.

Let $x = \sqrt{3\sqrt[3]{288\sqrt{3\sqrt[3]{288\sqrt{3\ldots}}}}}$ Then $x = \sqrt{3\sqrt[3]{288x}}$

This means that $x^2 = 3\sqrt[3]{288x}$ and $x^6 = 27(288x)$ or $x^6 = 6^5 x$

Since x is not zero, x must be 6.

S91SA22. Note that $x^3\left(x - \frac{1}{y^3}\right) - y^4\left(x - \frac{1}{y^3}\right) = 0$.

Thus, $(x^3 - y^4)(x - y^{-3}) = 0$ which implies that $x^3 = y^4$ or $x = y^{-3}$

and $\log_y x = -3$ or $4/3$.

S91S23. Let a number x be represented as I + f where I is [x] (the "Integer part" and f is the "fractional part" of x. Note that [2x] = 2I OR 2I + 1, dpending on whether f < 1/2 or f ≥ 1/2.
Also note that [3x] = 3I, 3I + 1, or 3I + 2, depending on whether 0 ≤ f < 1/3, 1/3 ≤ f < 2/3, or 2/3 ≤ f < 1.

Thus, [2x] + [3x] =

5I	if f < 1/3,
5I + 1	if 1/3 ≤ f < 1/2
5I + 2	if 1/2 ≤ f < 2/3
5I + 3	if 2/3 ≤ f < 1.

Thus, all numbers, except those congruent to 4(mod5) are obtainable. These include 1,2,3,5,6,7,8,10,11,12,13,15..., 98 which includes 79 numbers. Thus the answer is 79.

S91SA24. The number of flips in the first experiment can theoretically have values 1,2,3, ... If the first experiment requires "a" flips, then we wish the second experiment to require
3a, 3a+1, 3a+2, ... flips. Thus the answer is the sum:

$$\sum_{a=1}^{\infty}\sum_{b=3a}^{\infty}\left(\frac{1}{2}\right)^a\left(\frac{1}{2}\right)^b = \sum_{a=1}^{\infty}\left(\frac{1}{2}\right)^a\sum_{b=3a}^{\infty}\left(\frac{1}{2}\right)^b$$

$$= \sum_{a=1}^{\infty}\left(\frac{1}{2}\right)^a \cdot \frac{\left(\frac{1}{2}\right)^{3a}}{1-\frac{1}{2}} = \sum_{a=1}^{\infty}\left(\frac{1}{2}\right)^{4a-1} = \frac{\left(\frac{1}{2}\right)^3}{1-\left(\frac{1}{2}\right)^4} = \frac{2}{15}$$

SOLUTIONS

S91SA25. Use replacement to get :

$$2^{x} = \sqrt{2\sqrt[3]{2\sqrt[4]{2(2^{x})}}}.\quad \text{Then } 2^{2x} = 2\sqrt[3]{2\sqrt[4]{2^{x+1}}}$$

$$2^{2x-1} = \sqrt[3]{2\sqrt[4]{2^{x+1}}} \text{ so that } 2^{6x-3} = 2\sqrt[4]{2^{x+1}}$$

$$2^{6x-4} = \sqrt[4]{2^{x+1}} \text{ so that } 2^{24x-16} = 2^{x+1}.$$

We can now equate exponents to get:

$24x - 16 = x + 1$ so that $\boxed{x = 17/23.}$

S91SA26. $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$
Thus if $z^4 + z^3 + z^2 + z + 1 = 0$, then $z^5 - 1 = 0$ so $\boxed{z^5 = 1.}$

S91SA27. Divide by 8^y to obtain $3^x = 2^{7x-y+1} + 2^{2x-1}$. Since the right side is even unless one of the exponents is 0, either $y=7x+1$ or $2x-1=0$. The latter would not give integral x so $y=7x+1$. With this assumption we have $3^x = 1+2^{2x-1}$. When $x=1$ and $x=2$ there is equality. For integral $x>2$, the right hand side is greater than the left. Thus the only answers are (1,8) and (2,15).

S91SA28. We know that $11f(x+1) + 5f(x^{-1}+1) = \log_{10} x$. Substitute x^{-1} for x to get $5f(x+1) + 11f(x^{-1}+1) = -\log_{10} x$. Solving this system of equations for $f(x+1)$ gives $f(x+1) = \frac{1}{6}\log_{10} x$. So $f(x) = \frac{1}{6}\log_{10}(x-1)$.

Now $f(6) + f(17) + f(126) = \frac{1}{6}(\log_{10} 5+\log_{10} 16+\log_{10} 125) = \frac{1}{6}\log_{10} 10^4$
$= \boxed{2/3.}$

S91SA29. If x is an integer, then the left hand side is $x + 5x + 25x + 125x = 156x$. Since $156(6) < 10^3 < 156(7)$, we know that the value of x that we seek is such that $6 < x < 7$. Let us express the fractional part of x in base5: $x = 6 + (.abcde\ldots)_5$, where the variables a, b, c,... are digits from {0, 1, 2, 3, 4}. Now
$[x]+ [5x] + [25x] + [125x] = 6 + (30+a) + (150+5a+b) + (750+25a+5b+c)$
$= 936 + 31a + 6b + c = 1000$. Thus we need $31a + 6b + c = 64$and the only possibility is a=2, b=0, c=2 because of the restrictions on the digits. Now the base 5 digits to the right of 0 can be anything - but let them be 0's to minimize x. Thus $x = 6+2/5+0/25+2/125 = \boxed{6\frac{52}{125}}$

S91SA30. There are 7 possibilities for the first three tosses: 3 which have one head, 3 which have two heads, and 1 which has three heads. There are 8 possibilities for the last three tosses. Thus, the sample space has 7(8) = 56 outcomes, all of which are equally likely. Now if one head appears in the first 3 tosses, we need two heads in the last three tosses for a total of 3(3) = 9 possible ways. If two heads appear in the first three tosses, then we need one head later for 3(3)= 9 possibilities. If three heads appear in the first three tosses, we need zero heads in the last three which can occur in 1(1) way. Thus the answer is $\frac{9 + 9 + 1}{56} = \boxed{\frac{19}{56}}$

May 7, 1991

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1991 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S91SA7	1 or 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$
	S91SA19	4825 or $2^{12}+3^{6}$
	S91SA22	4/3 and -3 or all real numbers
Senior B	S91B24	1,048,575 or $2^{20}-1$

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML