NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — SPRING 1991

Part I: 10 Minutes

S91J1. In right triangle ABC, hypotenuse $\overline{AB}$ has length 4. Semicircles are constructed on sides BC and AC. Compute the sum of the areas of these semicircles.

S91J2. X is a positive two digit integer and Y is the two digit integer obtained by reversing the digits of X. How many positive integers X are there such that the difference X - Y is a perfect square?

Part II: 10 Minutes — NYCIML CONTEST ONE — SPRING 1991

S91J3. A jar contains pennies, nickels, and dimes only. The number of nickels is at least one-third the number of pennies and at most one-tenth the number of dimes. If the number of nickels and pennies together is at least 795, how many dimes are we assured of finding in the jar?

S91J4. Find the integer x and the digit y such that

$253 + (11)(x) = \sqrt{2134y4}$. Express the answer in the form (x,y).

Part III: 10 Minutes — NYCIML CONTEST ONE — SPRING 1991

S91J5 Pyramid ABCDE has a square base ABCD and congruent edges $\overline{AE}$, BE, CE and DE. If the length of the altitude of this pyramid is 5 and the length of each side of the base is 10, compute the length of AE.

S91J6 Find all ordered triples (a,b,c) such that a,b and c are non-zero real numbers and

$$a + b = ab;$$
$$b + c = 2bc;$$
and
$$c + a = 4ac.$$

ANSWERS

1. 2π
2. 22
3. 1990
4. (19,4)
5. $5\sqrt{3}$ or equivalent
6. $\left(\frac{2}{3}, -2, \frac{2}{5}\right)$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER TWO SPRING 1991

Part I: 10 Minutes

S91J7. In $\triangle ABC$, $\overline{BD}$ is drawn so that D is on $\overline{AC}$ between A and C and AB = AD. Compute the degree measure of $\angle CBD$ if $m\angle ABC - m\angle ACB = 26$.

S91J8. Compute the value of $a + b^2 + c^3$ if

$$a^2 + b^2 + c^2 = 6a + 8b + 10c - 50.$$

Part II: 10 Minutes NYCIML CONTEST TWO SPRING 1991

S91J9. Compute all values of k such that the x-axis is tangent to the graph of $y = x^2 + kx + 2k + 5$.

S91J10. Compute the sum of all possible four digit positive integers formed by using each different prime digit exactly once.

Part III: 10 Minutes NYCIML CONTEST TWO SPRING 1991

S91J11. A triangle has vertices at (9,0), (0,6) and the origin. Compute the coordinates of the point of intersection of the three medians of the triangle.

S91J12. A circle with diameter $\overline{AB}$ has chords $\overline{AC}$ and $\overline{BD}$ which intersect at E. If AE = 8, EC =2 and $\overline{BD}$ is bisected by $\overline{AC}$, compute the area of the circle.

ANSWERS

7. 13° or 13	9. −2, 10	11. (3,2)
8. 144	10. 113322	12. 28π

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER THREE — SPRING 1991

Part I: 10 Minutes

S91J13. In $\triangle ABC$, altitude $\overline{CD}$ intersects $\overline{AB}$ at D, where D is between A and B. If AD = 1, BD = 5, <u>and</u> the perimeter of $\triangle ABC$ is 18, then compute the length of $\overline{CD}$.

S91J14. Compute the value of

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{1990^2}\right)$$

Part II: 10 Minutes — NYCIML CONTEST THREE — SPRING 1991

S91J15 Quadrilateral ABCD is inscribed in a circle with CD = BC = 2 and AB = DA = 4. Compute the area of the circle.

S91J16. Find all possible real values that t can have in the system:

$$\frac{ab}{a+b} = t\ ; \qquad \frac{2b}{b+2} = t^2\ ; \qquad \frac{2a}{a+2} = t^3.$$

Part III: 10 Minutes — NYCIML CONTEST THREE — SPRING 1991

S91J17. Two of the vertices of a triangle are at (1,2) and (3,5). The third vertex is on the x-axis. If the point of intersection of the medians is on the line $2x + 3y = 5$, find the <u>coordinates</u> of the third vertex.

S91J18. Find all ordered triples (x,y,z) that satisfy the following system of equations:

$$x + 2y + 3z = 20$$

$$3x - 2y + z = 20$$

$$x^2 + y^2 + z^2 = 8x - 2y + 12z - 53$$

ANSWERS

13. $2\sqrt{6}$ or equivalent
14. $\frac{1991}{3980}$
15. 5π
16. -1, 1
17. $(-7,0)$
18. $(4,-1,6)$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — SPRING 1991

SOLUTIONS

S91J1. Let BC = x and AC = y. The desired sum is $\frac{\pi}{2}\left(\frac{x}{2}\right)^2 + \frac{\pi}{2}\left(\frac{y}{2}\right)^2$ which is equal to $\frac{\pi}{8}\left(x^2 + y^2\right)$. From the Pythagorean Theorem we have $x^2 + y^2 = 16$, so the above sum becomes $\frac{\pi}{8}(16) = 2\pi$.

S91J2. Let $X = 10a + b$ so $Y = 10b + a$ and $X - Y = 9(a-b)$. Now for $9(a-b)$ to be a perfect square, we must have $a-b = 0, 1, 4,$ or 9.

<u>Case One:</u> $a-b = 0$

Since neither a nor b can be 0, $a-b = 0$ implies that $a=b$, so that $X = 11, 22, 33, \ldots 99$. (Nine solutions)

<u>Case Two:</u> $a-b = 1$

$a = b + 1 \longrightarrow X = 21, 32, 43, \ldots, 98$. (Eight solutions)

<u>Case Three:</u> $a-b = 4$

$a = b+4 \longrightarrow X = 51, 62, 73, 84, 95$. (Five solutions)

Case Four ($a-b = 9$) leads to no viable solutions.

Thus there are 22 solutions in all.

S91J3. The words of the problem translate to symbols as:

$$\frac{P}{3} \le N \le \frac{D}{10} \quad \text{and} \quad P + N \ge 795$$

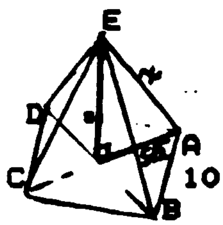
where P, N and D are the number of pennies, nickels and dimes in the jar, respectively. In order to involve the quantity P+N in the first inequality, add N/3 to both sides to get: $\frac{P}{3} + \frac{N}{3} \le N + \frac{N}{3}$ which simplifies to $P + N \le 4N$. Now inserting the fact that $P + N \ge 795$, we get $795 \le P + N \le 4N$ or $4N \ge 795$ which means that $N \ge 199$

From the first inequality, $N \le \frac{D}{10}$ which means that $D \ge 10N$ or $D \ge 1990$

Thus we are assured of finding 1990 dimes. (at least!)

S91J4. We have $(253 + 11x)^2 = 11^2(23+x)^2 = 2134y4$. In order for $2134y4$ to be divisible by 11, we must have $|2-1+3-4+y-4| = |y-4|$ divisible by 11. Since y is a decimal <u>digit,</u> y must be 4. Therefore, $11^2(23+x)^2 = 213444$ so that $(x+23)^2 = 1764$. This gives $x+23 = 42$ so that $x = 19$ ($x+23$ can not equal -42). Thus the answer is (19, 4).

S91J5.



$$(5)^2 + (5\sqrt{2})^2 = x^2$$
$$25 + 50 = x^2$$
$$x = \sqrt{75}$$
$$= 5\sqrt{3}$$

S91J6. Dividing both sides of the first equation by "ab" (which cannot be 0) yields $\frac{1}{a} + \frac{1}{b} = 1$. Likewise, dividing both sides of the second equation by "bc" and both sides of the third equation by "ca" gives, respectively,

$\frac{1}{b} + \frac{1}{c} = 2$ and $\frac{1}{a} + \frac{1}{c} = 4$

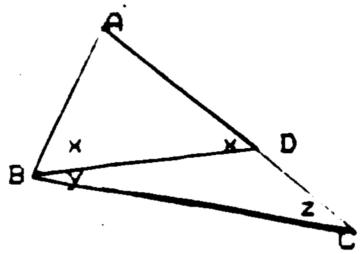
To solve, let $x = 1/a$, $y = 1/b$ and $z = 1/c$, and substitute in the three equations, giving $x+y=1$, $y+z=2$ and $z+x=4$.

Solving these simultaneously, gives $x = 3/2$, $y = -1/2$ and $z = 5/2$.

Thus, $a = 2/3$, $b = -2$ and $c = 2/5$. Answer: (2/3, -2, 2/5)

SOLUTIONS

S91J7.



Let $x = m\angle ABD = m\angle ADB$. $m\angle BDC = 180-x$. Letting $y = m\angle DBC$, $z = m\angle DCB$ and using an exterior angle of $\triangle BCD$ we get $\underline{x = z + y.}$ We were given $m\angle ABC - m\angle ACB = 26$ which means $x+y-z = 26$ or $x + y = z + 26$. This means that $\underline{x = z - y + 26}$. Adding the two underlined equations, gives $2x = 2z + 26$ so that $x = z + 13$. This, with the first equation, gives $y = 13$.

S91J8. $a^2 + b^2 + c^2 = 6a + 8b + 10c - 50.$
$a^2 - 6a + b^2 - 8b + c^2 - 10c = -50.$
$a^2 - 6a + 9 + b^2 - 8b + 16 + c^2 - 10c + 25 = 0.$
$(a-3)^2 + (b-4)^2 + (c-5)^2 = 0.$

The only way the above can happen is for $a = 3$, $b = 4$ and $c = 5$. This means that $a + b^2 + c^3 = 3 + 16 + 125 = 144$

S91J9. The graph of $y = x^2 + kx + 2k + 5$ is a parabola. In order for a parabola to be tangent to the x-axis, the discriminant must be zero. Thus we have $k^2 - 8k - 20 = 0$. This implies that $k = -2$ or 10.

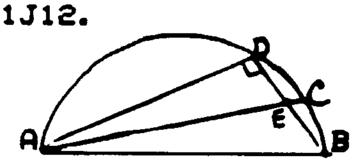
S91J10. The only prime digits are 2, 3, 5 and 7. There are $4! = 24$ positive four digit integers using each of these digits exactly once.
To find the sum of these twenty four integers, we can list all of them and add, or more easily realize the following:

1) Exactly SIX of the 24 integers end in 2.
2) Exactly SIX of the 24 integers end in 3.
3) Exactly SIX of the 24 integers end in 5.
4) Exactly SIX of the 24 integers end in 7.

Likewise, exactly SIX of the 24 integers have a "2" in the tens' place; exactly SIX have a "3" in the tens' place; exactly SIX have a "5" in the tens' place and exactly SIX have a "7" in the tens' place.
The same is true for the hundreds' and thousands' places. Thus the sum of the 24 integers is $6(2 + 3 + 5 + 7)(1 + 10 + 100 + 1000) = 6(17)(1111) = 102(1111)$ or 113322.

S91J11. The median with endpoints (9,0) and (0,3) has equation $y = \frac{-1}{3}x + 3$. The median with endpoints (0,0) and $(\frac{9}{2}, 3)$ has equation $y = \frac{2}{3}x$. [The third median is not needed!] These lines intersect at (3,2).
[Note that in general, a triangle with vertices. (x_1, y_1), (x_2, y_2), and (x_3, y_3) has centroid $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.]

S91J12.



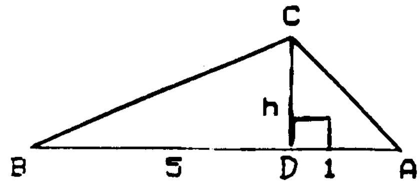
Since $(AE)(EC) = (BE)(ED)$, we have $BE = ED = 4$. Notice that since AB is a diameter, $\angle ADE$ is a right angle so we can use the Pythagorean Theorem to find AD. $AD^2 + ED^2 = AE^2$; so $AD^2 + 16 = 64$, giving $AD = \sqrt{48}$. Using the Pythagorean Theorem again, for $\triangle ADB$, we get $AB^2 = 48 + 64$, so that $AB^2 = 112$. The area of the circle is $\left(\frac{AB^2}{4}\right)\pi = 28\pi$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER THREE — SPRING 1991

SOLUTIONS

S91J13.



$AC = \sqrt{h^2+1}$; $BC = \sqrt{h^2+25}$

Thus, $\sqrt{h^2+1} + \sqrt{h^2+25} = 12$

This is equivalent to

$2h^2 + 26 + 2\sqrt{(h^2+1)(h^2+25)} = 144$ which gives $\sqrt{(h^2+1)(h^2+25)} = 59 - h^2$. Squaring both sides, we get $(h^2+1)(h^2+25) = h^4 - 118h^2 + 59^2$. This reduces to $144h^2 = 3456$ or $h^2 = 24$ or $h = 2\sqrt{6}$.

S91J14. Note that each factor is a difference of perfect squares, so we can use the fact that $1 - a^2 = (1-a)(1+a)$ in each case, giving:

$$\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \quad \ldots \quad \frac{1988}{1989} \cdot \frac{1990}{1989} \cdot \frac{1989}{1990} \cdot \frac{1991}{1990}$$

***** ***** ***** ... ******* ***********

The asterisks pair off terms whose product is one, leaving as our answer: $\frac{1}{2} \cdot \frac{1991}{1990} = \frac{1991}{3980}$

S91J15. Diagram 1 shows Quadrilateral ABCD with radii to A, B, C and D drawn. Note that Rectangle AB'C'D' in Diagram 2, drawn by interchanging triangles BOC and BOA, has the same area. Since its diagonal has length $2\sqrt{5}$, the radius of the circleis $\sqrt{5}$. Thus the area of the circle is 5π.

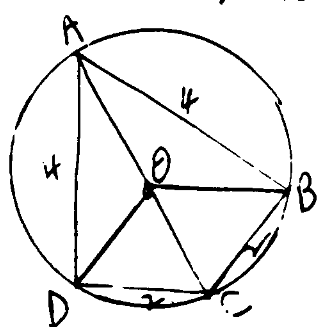


Diagram 1

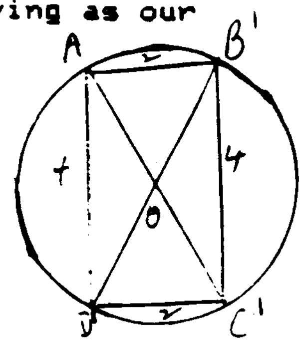


Diagram 2

S91J16. Note that if $a = 0$, then the first equation gives $t=0$ also and the second equation gives $b=0$. However, from the first equation, it can be seen that a and b CANNOT both be zero and hence t cannot be zero. A similar argument follows if $b = 0$.

Since none of these values is zero, it is certainly allowable to take reciprocals. Taking the RECIPROCALS of the three equations gives three new equations, easier to deal with; more specifically:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{t} \quad ; \quad \frac{1}{b} + \frac{1}{2} = \frac{1}{t^2} \quad ; \quad \frac{1}{a} + \frac{1}{2} = \frac{1}{t^3}$$

Adding the last two equations gives: $\frac{1}{a} + \frac{1}{b} + 1 = \frac{1}{t^2} + \frac{1}{t^3}$

Substituting the value of $\frac{1}{a} + \frac{1}{b}$ gives $\frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - 1 = 0$

This is equivalent to $\left(\frac{1}{t^2} - 1\right)\left(\frac{1}{t} + 1\right) = 0$. This gives $t = -1$ or 1.

S91J17. Let the third vertex be at $(k,0)$. Thus the coordinates of the centroid are $\left(\frac{k+4}{3}, \frac{7}{3}\right)$. In order for this to be on the line, we must have $\frac{2}{3}(k+4) + 7 = 5$ so that $k = -7$ and the coordinates of the third vertex are $(-7,0)$.

S91J18. In the last equation, transpose the variables to the left and complete the squares to get:
$(x-4)^2 + (y+1)^2 + (z-6)^2 = -53+16+1+36 = 0$. Thus the only potential solution is $x = 4$, $y = -1$ and $z = 6$. This solution satisfies the entire system and is therefore the answer $(4,-1,6)$.