NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER ONE — FALL 1990

Part I: 10 Minutes

F90S1. What is the prime factorization of $13^3 + 7^3$?

F90S2. A wooden cube of volume sixty-four cubic inches is painted on all six faces. It is then cut into sixty-four cubes, each with dimensions 1 inch x 1 inch x 1 inch. Find the number of unit cubes with exactly 2 faces painted.

Part II: 10 Minutes — NYCIML CONTEST ONE — FALL 1990

F90S3. If [ ] denotes greatest integer, solve for all real x:

$$\left[\frac{3x}{2}\right] - 1 = 5$$

F90S4. If $2f(x) + f\left(\frac{1}{x}\right) = \frac{x}{2}$, where x is any non-zero real number, compute f(3).

Part III: 10 Minutes — NYCIML CONTEST ONE — FALL 1990

F90S5. Find x if $\text{Arcsin}\left(\frac{12}{13}\right) + \text{Arcsin}\left(\frac{24}{25}\right) = \text{Arcsin}\ x$

F90S6. An apartment house has three mailboxes. The substitute mail carrier is in a rush and does not care in which boxes the mail is placed. If the carrier decides that each box should have at least ONE piece of mail, compute the number of ways this can be done if five pieces of mail are to be delivered.

ANSWERS

1. $5(2^2)(127)$
2. 24
3. $4 \leq x < \frac{14}{3}$
4. 17/18
5. $\frac{204}{325}$
6. 150

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER TWO — FALL 1990

Part I: 10 Minutes

F90S7. The sides of a triangle have lengths $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{7}$. Compute the area of this triangle and write your answer in simplest radical form.

F90S8. Suppose that $x + \frac{1}{x} = i$, where i is the imaginary unit. Compute the value of $x^4 + \frac{1}{x^4}$.

Part II: 10 Minutes — NYCIML CONTEST TWO — FALL 1990

F90S9. Find the ordered pair (a,b) which satisfies the following:

$$\sum_{x=1}^{10} (ax^2 + b) = \sum_{x=11}^{20} (bx + a) = 11915$$

F90S10. Find a simple fraction with rational denominator equivalent to

$$\frac{1}{\sqrt[3]{7} - 1}.$$

Part III: 10 Minutes — NYCIML CONTEST TWO — FALL 1990

F90S11. If a and b represent digits, find the ordered pair (a,b) such that 17171717ab is divisible by 99.

F90S12. A single die is thrown until a six occurs. What is the probability that an even number of throws is needed?

ANSWERS

7. $\frac{\sqrt{26}}{2}$ or equivalent

8. 7

9. (29,75)

10. $\frac{\sqrt[3]{49} + \sqrt[3]{7} + 1}{6}$

11. (3,1)

12. $\frac{5}{11}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER THREE — FALL 1990

Part I: 10 Minutes

F90S13. Compute the value of $\log_2(\log_{16}4)^{\log_5 125}$.

F90S14. A wooden cube of volume n^3 cubic inches is painted and then cut into n^3 cubes. Compute the value of n if there are 180 unit cubes with exactly 2 faces painted.

Part II: 10 Minutes — NYCIML CONTEST THREE — FALL 1990

F90S15. Find the coefficient of x^{499} in expanding the product
$$(x-2)(x-4)(x-6)(x-8)\ldots(x-1000)$$

F90S16. If [ ] denotes greatest integer, solve for all x:
$$2[2x]^2 - 9[2x] + 9 = 0.$$

Part III: 10 Minutes — NYCIML CONTEST THREE — FALL 1990

F90S17. In quadrilateral ABCD, CD = 1, BC = 2, m∠BCD = 120 and the angles at D and B are right. Find AB.

F90S18. Suppose $x + \frac{1}{x} = -1$ and $f(p) = x^p + \frac{1}{x^p}$. Compute the value of
$$\sum_{n=1}^{1990} f(2^n).$$

ANSWERS

13. -3

14. 17

15. -250500

16. $\frac{3}{2} \le x < 2$

17. $\frac{4\sqrt{3}}{3}$ or equivalent

18. -1990

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER FOUR — FALL 1990

Part I: 10 Minutes

F90S19. Compute the value of $\cot\left(\operatorname{Arcsin}\left(\frac{1}{\pi}\right) + \operatorname{Arccos}\left(\frac{1}{\pi}\right)\right)$

F90S20. Hexagon ABCDEF is inscribed in a circle with AB = CD = EF = 2 and BC = DE = FA = 10. Compute the area of an equilateral triangle inscribed in this circle.

Part II: 10 Minutes — NYCIML CONTEST FOUR — FALL 1990

F90S21. If $x + \frac{1}{x} = 10$, compute the absolute value of $x - \frac{1}{x}$.

F90S22. How many distinct terms will there be if $(x + y + z)^{17}$ is expanded algebraically and simplified?

Part III: 10 Minutes — NYCIML CONTEST FOUR — FALL 1990

F90S23. Compute f(2) if $2f(x) + \sqrt{2}f\left(\frac{1}{x}\right) = 2^x$ for all non-zero real x.

F90S24. A coin is flipped until two tails have occured. If the first tail occured on flip "a" and the second on flip "b" then what is the probability that b = 4a? (Note the answer should not involve a or b.)

ANSWERS

19. 0
20. $31\sqrt{3}$
21. $4\sqrt{6}$ or equivalent
22. 171
23. 3
24. 1/15

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION — CONTEST NUMBER FIVE — FALL 1990

Part I: 10 Minutes

F90S25. If $16(\log x)^2 + 9(\log y)^2 = 24(\log x)(\log y)$, find y in terms of x.

F90S26. Compute the number of radians in $f(1) + f(2) + f(3)$ if $f(x) = \text{Arc tan}\left(\frac{1}{x}\right)$.

Part II: 10 Minutes — NYCIML CONTEST FIVE — FALL 1990

F90S27. An equiangular octagon has sides which have lengths that alternate between 1 and 2. It can be shown that a circle can be circumscribed about this octagon. If the area of the circumscribing circle is $\pi(a\sqrt{2}+b)$, find the ordered pair (a, b).

F90S28. If [ ] denotes greatest integer, find all real solutions of $[2x]^2 = x + \frac{3}{2}$.

Part III: 10 Minutes — NYCIML CONTEST FIVE — FALL 1990

F90S29. Compute f(4) given that $2f(x^2) + 3f(19-5x) = x^3$ for all real numbers x.

F90S30. Al has a die with six faces. The numbers 1, 2, 3 and 4 appear on four faces, while blanks appear on two of them. If a blank shows up, he just rolls again until he gets a 1, 2, 3 or 4. Joyce has a normal die with six faces showing the numbers 1 through 6. Scott has a die with 12 faces and the numbers from 1 to 12 showing. Al, Joyce and Scott take turns rolling their dice in that order, until someone rolls a "1" and wins. If these people are using "fair" dice, that is, the n integers from 1 to n are equally likely to appear, compute the probability that Scott will win.

ANSWERS

25. $y = x^{4/3}$ or equivalent
26. $\frac{\pi}{2}$
27. $\left(1, \frac{5}{2}\right)$
28. $x = \frac{-1}{2}$
29. 13
30. $\frac{5}{41}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION — CONTEST NUMBER ONE — FALL 1990

SOLUTIONS

F90S1. Note that $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
Letting $x = 13$ and $y = 7$, we get:
$13^3 + 7^3 = (13+7)(169 - 91 + 49) = (20)(127) = 5(2^2)(127)$

F90S2. In order to have two faces painted, the cube must be on an edge, but not be a corner cube. In general, a cube consisting of n^3 unit cubes will have $12(n-2)$ cubes with two faces painted since there are twelve edges. Thus $12(n-2) = 12(4-2) = 24$.

F90S3.

$$\left[\frac{3x}{2}\right] - 1 = 5 \longrightarrow \left[\frac{3x}{2}\right] = 6$$

The only way for this to happen is for $6 \le \frac{3x}{2} < 7$

or $12 \le 3x < 14 \longrightarrow 4 \le x < \frac{14}{3}$

F90S4. Let $x = 3 \longrightarrow 2f(3) + f\left(\frac{1}{3}\right) = \frac{3}{2}$

Let $x = \frac{1}{3} \longrightarrow f(3) + 2f\left(\frac{1}{3}\right) = \frac{1}{6}$

Solve the two simultaneously, get $f(3) = 17/18$

F90S5. Let $A = \text{Arcsin}\left(\frac{12}{13}\right)$ and let $B = \text{Arcsin}\left(\frac{24}{25}\right)$ Substituting in the given equation,

gives $x = \sin(A+B) = \sin A \cos B + \cos A \sin B$

Needed: $\cos A$ and $\cos B$

Using a "5-12-13 $\triangle$" we get $\cos A = \frac{5}{13}$

Using a "7-24-25 $\triangle$" we get $\cos B = \frac{7}{25}$

Thus $x = \frac{12}{13} \cdot \frac{7}{25} + \frac{24}{25} \cdot \frac{5}{13} = \frac{204}{325}$

F90S6. For each mailbox to be used, there are two cases to consider:
CASE ONE: One mailbox gets three letters and the remaining two get one each. Think of doing this by first choosing the three letters that go into one box; then choosing a letter that goes into a box by itself; and then taking the remaining letter for the last mailbox. Last of all, match the three sets with the three mailboxes. The number of ways of doing this is $({}_5C_3)({}_2C_1)({}_1C_1)\frac{3!}{2!} = (10)(2)(1)(3) = 60$

CASE TWO: Two mailboxes get two letters each and the remaining box gets one. Using an analysis comparable to that above, we get
$({}_5C_2)({}_3C_2)({}_1C_1)\frac{3!}{2!} = (10)(3)(1)(3) = 90$

Since these are the only possible cases, the total number of ways the mail can be delivered is $60 + 90 = 150$.

SOLUTIONS

F90S7. Let x be the measure of the angle between the two smaller sides. The law of cosines gives: $\cos x = \dfrac{5 + 6 - 7}{2\sqrt{30}} = \dfrac{2}{\sqrt{30}}$

Using $\sin^2 x + \cos^2 x = 1$ we get $\sin x = \sqrt{\dfrac{13}{15}}$. Thus the area is $\dfrac{1}{2}\sqrt{30}\cdot\sqrt{\dfrac{13}{15}} = \dfrac{\sqrt{26}}{2}$

F90S8. $\left(x + \dfrac{1}{x}\right)^2 = x^2 + \dfrac{1}{x^2} + 2 = i^2 = -1$

Thus $x^2 + \dfrac{1}{x^2} = -3$ which means $\left(x^2 + \dfrac{1}{x^2}\right)^2 = 9$

$x^4 + \dfrac{1}{x^4} + 2 = 9$ so that $x^4 + \dfrac{1}{x^4} = 7.$

F90S9. $$\sum_{x=1}^{10}(ax^2 + b) = 10b + a\cdot\frac{(10)(11)(21)}{6} = 10b + 385a$$

[Note the above uses $1^2+2^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$]

$$\sum_{x=11}^{20}(bx + a) = 10a + b\cdot\left[\frac{(20)(21)}{2} - \frac{(10)(11)}{2}\right] = 10a + 155b$$

Thus, solving $385a + 10b = 10a + 155b = 11915$, we get $a = 29$, $b = 75$. Answer: (29, 75).

F90S10. Since $a^3-1 = (a-1)(a^2+a+1)$, multiply the numerator and denominator by

$\sqrt[3]{49} + \sqrt[3]{7} + 1$, gives $\dfrac{\sqrt[3]{49} + \sqrt[3]{7} + 1}{6}$

F90S11. Simple division could be used but perhaps a safer technique is to use the divisibility rules for 9 and 11.

<u>To be divisible by 9:</u> $1 + 7 + 1 + 7 + 1 + 7 + 1 + 7 + a + b = 32 + a+b$ must be a multiple of 9. Since 32 is congruent to 5 (mod 9) we can simplify the above to: <u>a+b+5 is divisible by 9.</u>

<u>To be divisible by 11:</u> $1 - 7 + 1 - 7 + 1 - 7 + 1 - 7 + a-b = -24 + a-b$ must be a multiple of 11. Since -24 is congruent to 9 (mod 11), we can simplify the above to: <u>a-b+9 is divisible by 11.</u>

Since a and b are <u>digits</u>, the first criterion gives $a+b+5 = 9$ or 18. This means that $a+b = 4$ or 13. The second criterion gives $a-b+9 = 11$ or 0. (Note that $a-b+9$ will be at most 18!) This means that $a-b = 2$ or -9. Combining these, we get $2a = -5$, 4, 6, or 15 , of which, a can only be 2 or 3. If $a = 2$, there is no solution for b. If $a = 3$, then $b = 1$. Hence, the answer is (3, 1)

F90S12. We can add probabilities of mutually exclusive events. We need P(first 6 on throw #2)+P(first 6 on #4) +P(first 6 on #6)+...

$= \dfrac{(5)(1)}{6^2} + \dfrac{(5)^3(1)}{6^4} + \dfrac{(5)^5(1)}{6^6} + \dfrac{(5)^7(1)}{6^8} + \ldots$ which is a geometric series with sum 5/11.

SOLUTIONS

F90S13. $\log_2(\log_{16}4)^{\log_5 125} = \log_2(1/2)^{\log_5 125} = \log_2(1/2)^3 = -3$

F90S14. In order to have two faces painted, the cube must be on an edge, but not be a corner cube. In general, a cube consisting of n^3 unit cubes will have $12(n-2)$ cubes with two faces painted. Thus $12(n-2) = 180$ which gives $n = 17$.

F90S15. This product has 500 factors. Terms with x^{499} include $-2x^{499}, -4x^{499}, -6x^{499}, \ldots, -1000x^{499}$
The sum of these coefficients is $-2-4-6-8-\ldots-1000$. Using the formula for the sum of an arithmetic series, this is

$$\frac{500}{2}(-2-1000) = -250500$$

F90S16. Factor to get $(2[2x] - 3)([2x] - 3) = 0$
This implies that $[2x] = \frac{3}{2}$ or 3. Now $[2x]$ must be an integer, thus $[2x] = 3$ which implies that $3 \le 2x < 4$ or $\frac{3}{2} \le x < 2$.

F90S17. We know that $AB^2 + BC^2 = AC^2 = AD^2 + CD^2$.
Let $AB = x$ and $AD = y$. This gives $x^2 + 4 = y^2 + 1$ and $y^2 = x^2 + 3$.
Note that $m\angle BAD = 60$ and by the law of cosines,
$x^2 + y^2 - 2xy\cos 60 = BD^2 = 1^2 + 2^2 - 2(2)(1)\cos 120$ or
$x^2 + y^2 - xy = 7$. Since $y^2 = x^2 + 3$. and $y = \sqrt{x^2 + 3}$
This gives
$$4x^4 - 16x^2 + 16 = x^4 + 3x^2$$
$$3x^4 - 19x^2 + 16 = 0$$
$$(3x^2 - 16)(x^2 - 1)0 = 0.$$
Since $x > 0$, we get $x = 1$ or $\frac{4}{3}\sqrt{3}$

Substituting in $y = \sqrt{x^2+3}$ we get $(x,y) = (1,2)$ or $(\frac{4}{3}\sqrt{3}, \frac{5}{3}\sqrt{3})$

Since (1,2) does not satisfy our second equation, the only answer is $\frac{4}{3}\sqrt{3}$.

F90S18. Suppose $f(2^n) = x^{2^n} + \dfrac{1}{x^{2^n}} = T$,

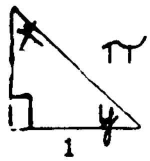
then $f(2^{n+1}) = x^{2^{n+1}} + \dfrac{1}{x^{2^{n+1}}} = \left[x^{2^n} + \dfrac{1}{x^{2^n}}\right]^2 - 2 = \left[f(2^n)\right]^2 - 2$

$= T^2 - 2$. Since $f(2^0) = -1$ and $f(2) = -1$,

$f(2^2) = (-1)^2 - 2 = -1$, etc. Thus, $f(2^n) = -1$ for all positive integral n. Thus the answer is -1990.

SOLUTIONS

F90S19.



Draw the indicated right triangle. Now, $\sin x = \dfrac{1}{\pi}$ and $\cos y = \dfrac{1}{\pi}$.

$x = \text{Arc}\sin\dfrac{1}{\pi}$ and $y = \text{Arc}\cos\dfrac{1}{\pi}$

Since $x + y = 90°$, the answer is $\cot 90° = 0$

F90S20. Since $\triangle$ ACE is equilateral, m∠ABC = 120. Thus, using the law of cosines, we have $(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos 120°$. This equals $4 + 100 - 2(20)\left(\dfrac{-1}{2}\right) = 124$. The area sought is $\dfrac{(AC)^2\sqrt{3}}{4}$

This gives $31\sqrt{3}$.

F90S21. $\left(x + \dfrac{1}{x}\right)^2 = x^2 + \dfrac{1}{x^2} + 2 = 100 \Longrightarrow x^2 + \dfrac{1}{x^2} - 2 = 96$

Thus $\left(x - \dfrac{1}{x}\right)^2 = 96 \Longrightarrow \left|x - \dfrac{1}{x}\right| = \sqrt{96} = 4\sqrt{6}$ or equivalent

F90S22. $(x+y+z)^{17} = x^{17} + {}_{17}C_1 x^{16}(y+z) + {}_{17}C_2 x^{15}(y+z)^2 + \dots + {}_{17}C_{16}\, x(y+z)^{16} + (y+z)^{17}$

Now $(y+z)^k$ has k+1 distinct terms, so the answer is $1 + 2 + 3 + \dots + 17 + 18 = 171$

F90S23. Let $x = 2$: $2f(2) + \sqrt{2}f\left(\dfrac{1}{2}\right) = 4$

Let $x = \dfrac{1}{2}$: $\sqrt{2}f(2) + 2f\left(\dfrac{1}{2}\right) = \sqrt{2}$ Solving gives $f(2) = 3$

F90S24. We need to obtain an initial tail in "a" flips then obtain a second tail in 3a flips. The probability that this happens is $\left(\dfrac{1}{2}\right)^a\left(\dfrac{1}{2}\right)^{3a} = \left(\dfrac{1}{2}\right)^{4a}$. Since "a" can be any of 1,2,3... find the sum:

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{4k}$$ which is geometric giving $\dfrac{1}{15}$

SOLUTIONS

F90S25. $16(\log x)^2 + 9(\log y)^2 = 24 \log x \log y$

$16(\log x)^2 - 24 \log x \log y + 9(\log y)^2 = 0$

$(4 \log x - 3 \log y)^2 = 0 \longrightarrow x^4 = y^3$

Thus $y = \sqrt[3]{x^4} = x^{4/3}$

F90S26. Let $A = \tan x$ and $B = \tan y$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{A + B}{1 - AB}$

-or- $\text{Arctan}\dfrac{A + B}{1 - AB} = x + y = \text{Arctan}A + \text{Arctan}B$

Thus we have the famous formula: $\text{Arctan}A + \text{Arctan}B = \text{Arctan}\left(\dfrac{A + B}{1-AB}\right)$

Using this, we get $f(1)+f(2)+f(3) = \text{Arctan}1 + \left(\text{Arctan}\frac{1}{2} + \text{Arctan}\frac{1}{3}\right)$

$= \dfrac{\pi}{4} + \text{Arctan}\left(\dfrac{5/6}{1 - 1/6}\right) = \dfrac{\pi}{2}$

F90S27. Let A, B, C be three consecutive vertices with AB = 1, BC = 2. Now $m\angle ABC = 135°$ and $m\widehat{AC} = 90°$. By the law of cosines, $AC^2 = 1^2 + 2^2 - 2(1)(2)\cos 135° = 5 + 2\sqrt{2}$. The radius of the circle is $\dfrac{AC}{\sqrt{2}} = \sqrt{\dfrac{5}{2} + \sqrt{2}}$ and the area is $\pi(\sqrt{2} + \frac{5}{2})$. Thus the answer $(1, \frac{5}{2})$ (or equivalent)

F90S28. Since $[2x]^2$ must be an integer, $x + \frac{3}{2}$ must also be an integer. Let $x = m/2$ where m is an odd integer (in order for $x + 3/2$ to be integral). Substituting in the equation, we get $[m]^2 = \dfrac{m+3}{2}$ or $m^2 = \dfrac{m+3}{2} \longrightarrow 2m^2 - m - 3 = 0$

$(m+1)(2m-3) = 0 \longrightarrow m = \frac{3}{2}$ or $m = -1$. This means $x = \frac{3}{4}$ or $x = \frac{-1}{2}$

Since $\frac{3}{4}$ does NOT check the original, the only solution is $x = \frac{-1}{2}$ (or equivalent)

F90S29. This problem differs from F90S23 because f(x) cannot easily be found. Let x = 2: $2f(4) + 3f(9) = 8$. Let x = 3: $2f(9) + 3f(4) = 27$. Solve for f(4) to obtain f(4) = 13.

F90S30. If Scott is to win on his first turn. Al does not roll a "1" neither does Joyce, but Scott does. The probability of this happening is $\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{12}$

If Scott is to win on his second turn. Al does not roll a "1" on either of his first two turns, neither does Joyce. Scott must throw something other than a "1" the first time and a "1" the second time. The probability of this happening is $\left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{11}{12}\right)\cdot\left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{12}\right)$

Continuing this way, the probility that Scott wins is:

$\left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{12}\right) + \left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{11}{12}\right)\cdot\left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{12}\right) + \left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{11}{12}\right)^2\cdot\left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{12}\right) + \ldots$

This is geometric with $a = \frac{5}{96}$ and $r = \frac{55}{96}$ and $S = \frac{5}{41}$