NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER ONE — FALL 1990

Part I: 10 Minutes

F90J1. Compute the value of: $\frac{1}{1991} + \frac{1992 \times 1990}{1991} - 1992$

F90J2. In quadrilateral ABCD, $\overline{AD} \perp \overline{CD}$ and $\overline{AB} \perp \overline{CB}$. If BC = 1, CD = 2, and AD = 3, what is the length of $\overline{AB}$?

Part II: 10 Minutes — NYCIML CONTEST ONE — FALL 1990

F90J3. If $\frac{58}{15} = 3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{x}}}$ compute the value of x.

F90J4. In triangle ABC, AC = 5, CB = 12 and AB = 13. The bisector of $\measuredangle$CAB intersects CB at D. Compute the length of CD.

Part III: 10 Minutes — NYCIML CONTEST ONE — FALL 1990

F90J5. Consider the integer 1A7B where A and B are digits. If A and B are chosen at random, what is the probility that the resulting four digit integer is a multiple of 4?

F90J6. Compute the sum of ALL of the digits of the EVEN integers between 1 and 1001?

ANSWERS

1. -1
2. $2\sqrt{3}$ or equivalent
3. 2
4. 10/3 or equivalent
5. 0.2 or equivalent
6. 6501

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER TWO — FALL 1990

Part I: 10 Minutes

F90J7. A pizza parlor offers plain pizza slices and pizza slices with toppings. The toppings can be pepperoni, mushrooms, broccoli, onions, and meatbealls. You can have as many toppings as you like, if any at all. If a pizza lover buys a <u>different</u> type of slice everyday, how many days will it take to have every possible type of pizza?

F90J8. In pentagon ABCDE, $\overline{AE} \perp \overline{AB}$ <u>and</u> $\overline{BC} \perp \overline{CD}$. F is the foot of the perpendicular from B to DE. If AB = 3, BC = 4, CD = 5, DF = 6, and EF = 7, what is the length of $\overline{AE}$?

Part II: 10 Minutes — NYCIML CONTEST TWO — FALL 1990

F90J9. A student thought that $\sqrt{5\frac{2}{3}} = 5\sqrt{\frac{2}{3}}$. (This is certainly not true!) Find all integers that may replace the "5" in the equation so that the resulting equality is true.

F90J10. Find all ordered pairs of real numbers (a,b) which satisfy $(a + b)^2 = (a + 1)(b - 1)$

Part III: 10 Minutes — NYCIML CONTEST TWO — FALL 1990

F90J11. How many seven digit positive integers whose five right-most digits are 12345 are multiples of 11?

F90J12. Alice, Betty and Charles sit at a circular table with fourteen other people. If the people sit randomly, what is the probability that Alice is next to either Betty or Charles or both?

ANSWERS

7. 32
8. $3\sqrt{5}$ or equivalent
9. 2
10. (-1,1)
11. 8
12. $\frac{29}{120}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 1990**

Part I: 10 Minutes

F90J13. Arthur takes twice as long as Bob to do a certain job. If they would take 10 hours to do the job working together, how long would it take Bob to do the job working alone?

F90J14. After two dice are tossed, ten faces can be seen. What is the probability that the sum of all spots that can be seen on those ten faces is divisible by 7?

Part II: 10 Minutes **NYCIML CONTEST THREE** **FALL 1990**

F90J15. Compute the sum of the digits of all multiples of 25 from 1 to 100,000 inclusive.

F90J16. What is the least positive three digit integer k such that

$$\frac{k^5 + k + 1}{k^2 + k + 1}$$ **is an integer.**

Part III: 10 Minutes **NYCIML CONTEST THREE** **FALL 1990**

F90J17. In $\triangle ABC$, $\overline{XP} \perp \overline{AB}$ and $\overline{XQ} \perp \overline{BC}$, $\overline{XR} \perp \overline{AC}$. X is in the interior of $\triangle ABC$, and P, Q and R are respectively on $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. If BP = 1, BQ = 2, CQ = 3, CR = 4 and AR = 5, find the length of AP.

F90J18. What is the smallest integer x for which:

$$\frac{1}{2} - \sqrt{x-19} + \sqrt{x-89} > 0 \ ?$$

ANSWERS

13.	15	15.	78001	17.	$\sqrt{15}$
14.	$\frac{1}{6}$	16.	100	18.	4955

SOLUTIONS

F90J1. $\frac{1}{1991} + \frac{1992 \times 1990}{1991} - 1992 = \frac{1}{1991} + \frac{1992(1990) - 1992(1991)}{1991}$

$\frac{1}{1991} + \frac{1992(1990-1991)}{1991} = \frac{1 + 1992(-1)}{1991} = \frac{-1991}{1991} = \boxed{-1}$

F90J2. Using the Pythagorean Theorem twice,

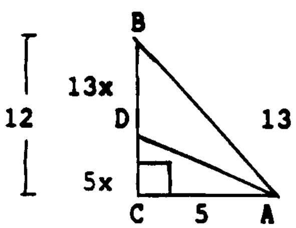
$$AB^2 + BC^2 = AC^2 = AD^2 + CD^2$$

Thus, $AB^2 + 1 = 9 + 4 \Longrightarrow AB = 2\sqrt{3}$ or equivalent.

F90J3. $\frac{58}{15} = 3 + \frac{13}{15} = = 3 + \cfrac{1}{\frac{15}{13}} = 3 + \cfrac{1}{1 + \frac{2}{13}}$

$= 3 + \cfrac{1}{1 + \cfrac{1}{\frac{13}{2}}} = 3 + \cfrac{1}{1 + \cfrac{1}{6 + \frac{1}{2}}}$ Thus $\boxed{x = 2}$

F90J4.



Using the angle bisector theorem, the two segments are proportional to the two sides of the angle. Thus $5x + 13x = 12 \Longrightarrow x = 2/3$

Thus $5x = \boxed{10/3}$ or equivalent.

F90J5. In decimal notation, the original number 1A7B is written as 1000 + 100A + 7B. Note that 7B is a <u>two-digit</u> integer. In order for the original number to be divisible by 4, we only need 7B to be divisible by 4. Thus B = 2 or 6. A can assume any digital value, so there are 10(2) or 20 possibilities for the pair A,B. There are 10(10) choices for the pair A,B. Thus the probability is 20/100 or 1/5 or $\boxed{0.2}$

F90J6. All even integers in the interval [0,998] can be listed by writing the list of two digit numbers 00 01 02 03 ... 99 and by placing THE NUMBERS 0, 2, 4, 6 and 8 at the end of each number on the list. The <u>original</u> list contains 200 digits, with each digit 0-9 appearing 200/10 = 20 times. The sum of the digits on this list is $20(0+1+2+3+...+9) = 900$

Adjoining the numbers 0,2,4,6,and 8 to the list means that we are considering the original list <u>FIVE</u> times. The sum so far is 5(900) or 4500. The additional numbers add $100(0+2+4+6+8) = 2000$ to the sum for a total of 6500. This sum does not include 1000. Adding this gives a total of

$\boxed{6501}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION | CONTEST NUMBER TWO | FALL 1990

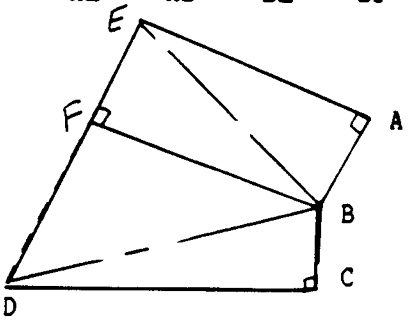
SOLUTIONS

F90J7. **Method 1:** Focus on the toppings. Each day this person makes several decisions. Should I take pepperoni or leave it out today? Should I take the mushrooms, or leave them out today ... and so on! Thus for each topping, there are two choices, giving $2^5 = \boxed{32}$ possible types.

Method 2: $_5C_0 + {_5C_1} + {_5C_2} + {_5C_3} + {_5C_4} + {_5C_5} = 1+5+10+10+5+1 = 32$

F90J8. From the Pythagorean Theorem,

$$BF^2 + DF^2 = BD^2 = BC^2 + CD^2 \Longrightarrow BF^2 + 36 = 16 + 25 \Longrightarrow BF^2 = 5$$
$$AE^2 + AB^2 = BE^2 = BF^2 + EF^2 \Longrightarrow AE^2 + 9 = 5 + 49 \Longrightarrow AE^2 = 45$$



Thus AE = $\boxed{\sqrt{45} \text{ or } 3\sqrt{5}}$

(Note: It does not matter if F actually falls between D and E.)

F90J9. $\sqrt{x + \frac{2}{3}} = x\sqrt{\frac{2}{3}} \Longrightarrow x + \frac{2}{3} = \frac{2x^2}{3} \Longrightarrow 2x^2 - 3x - 2 = 0$

$(2x+1)(x-2) = 0$ so that $x = -1/2$ (reject) or $\boxed{x = 2}$

F90J10. Expand and rearrange to obtain: $a^2 + (b+1)a + b^2 - b + 1 = 0$. Treat this as a quadratic in a and solve for a:

$a = \dfrac{-(b+1) \pm \sqrt{-3(b-1)^2}}{2}$. Since b is a real number, then the only way for a to also be real is if $b = 1$. This implies that $a = -1$. Thus the only solution is $\boxed{(-1,1)}$

F90J11. Let a possible solution be represented by ab12345. Using the divisibility test for 11, we need $|a-b+1-2+3-4+5| = |a-b+3|$ to be a multiple of 11, possibly zero. Since a and b are digits, with "a" not equal to zero, $-8 \le a-b \le 9$. We need a-b to be 8 or -3 in order for $|a-b+3|$ to be a multiple of 11. If a-b is 8, we have the ordered pairs (9,1) and (8,0). If a-b is -3, we have the ordered pairs (1,4), (2,5), (3,6), (4,7), (5,8) and (6,9). Thus there are <u>eight</u> such numbers ab12345 divisible by 11.

F90J12. Do the problem by considering the probability that Alice does not sit next to Betty or Charles. Let Alice sit down first. In order for Betty or Charles to sit away from Alice, there are 14 choices for the person on Alice's left and then 13 choices on her right. This gives 14(13)(14!) ways in which Alice does not sit next to Betty or Charles. Since there are (17-1)! ways to sit 17 people at a table, the answer to the question is

$$1 - \frac{14(13)(14!)}{16!} = 1 - \frac{14(13)}{16(15)} = 1 - \frac{91}{120} = \boxed{\frac{29}{120}}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER THREE FALL 1990

SOLUTIONS

F90J13. **<u>Method 1:</u>**
Let x = the amount of time required by Bob alone. Then:

$$\frac{1}{x}\text{ jobs/hour} + \frac{1}{2x}\text{ jobs/hour} = \frac{1}{10}\text{ jobs/hour} \quad\text{or}\quad \frac{3}{2x} = \frac{1}{10} \quad\text{or } x = 15$$

Thus Bob takes <u>15</u> hours alone.

<u>Method 2:</u> (METHOD OF FALSE POSITION) Assume B takes one hour and A takes 2 hours. Then the combined work rate is 1/1 + 1/2 = 3/2 jobs per hour. This exceeds the actual job rate of 1/10 jobs per hour by a factor of 3/2 divided by 1/10 or 15. Thus the actual time required by Bob is 15 hours.

F90J14. The sum of the spots on the twelve faces of two dice is 42. Thus, the sum of the ten faces that can be seen is divisible by seven, if and only if the sum of the spots on the two faces that are NOT showing (down) is divisible by 7. This probability is <u>1/6.</u>

F90J15. Consider the list 000, 001, 002, ..., 999. To this list we we will adjoin (at the end) the digits 00, 25, 50,and 75. In the list there are 3000 digits and each digit 0-9 occurs exactly 300 times. Thus the sum of the digits in the list is 300(0+1+2+3+...+9) = 13500. To each of the 1000 numbers on the list, add the digits 00, 25, 50 and 75. The sum is 4(13500) + 1000 [(0+0) + (2+5) + (5+0) + (7+5)] = 78000. Include 10^3 and the answer is <u>78001.</u>

F90J16. Using long division:

$$\begin{array}{r|l} & k^3 - k^2 + 1 \\ \hline k^2 + k + 1 & k^5 + \qquad\qquad k + 1 \\ & -(k^5 + k^4 + k^3) \\ \hline & -k^4 - k^3 + \quad k + 1 \\ & -(-k^4 - k^3 - k^2) \\ \hline & k^2 + k + 1 \end{array}$$

Thus the quotient is ALWAYS an integer <u>if k</u> is an integer. Therefore, the smallest three digit integer k is $\boxed{100}$.

F90J17. From the Pythagorean Theorem,

$$BP^2 + PX^2 = BQ^2 + QX^2$$
$$CQ^2 + QX^2 = CR^2 + RX^2$$
$$AR^2 + RX^2 = AP^2 + PX^2$$

Adding and subtracting like terms, we get:

$$AP^2 + CR^2 + BQ^2 = AR^2 + CQ^2 + BP^2$$
$$AP^2 + 16 + 4 = 25 + 9 + 1 \implies AP^2 = 15 \text{ so } AP = \boxed{\sqrt{15}}$$

F90J18.
$1/2 - \sqrt{x-19} + \sqrt{x-89} > 0 \implies 1/2 + \sqrt{x-89} > \sqrt{x-19}$ Since both sides of the ine<u>quali</u>ty are positive, <u>squar</u>e to obtain:
$1/4 + x - 89 + \sqrt{x-89} > x-19 \implies \sqrt{x-89} > 70-1/4$ Squaring again, we have $x-89 > 4900 - 35 + 1/16$. Thus the answer is 4900 +89-35 + 1 =

$\boxed{4955}$