NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE SPRING, 1990

Part I: 10 Minutes

S90B1. John counted the number of possible subsets that set X has. Mary counted the number of possible subsets that set Y has. John's number is 96 more than Mary's. How many <u>elements</u> does set X have?

S90B2. Point X is on side BC of rectangle ABCD. If AX = 5, XD = 12, and AD = 13, find the area of rectangle ABCD.

Part II: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90B3. Find the unique real number x such that

$$\sqrt{x\left(\sqrt[3]{x}\right)} = 64.$$

S90B4. One side of a very accurate ruler is to be marked in units of 1/9 inch and also in units of 1/24 inch. How many different marks must be made from the 1 inch mark to the 2 inch mark, including the two end points?

Part III: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90B5. A drawer is full of many easter eggs. Each egg is either red, white, or blue. An easter bunny reaches into the drawer, and without looking, begins to take out eggs. The bunny will stop when it gets three eggs of the same color. At worst, how many eggs will the bunny have to remove?

S90B6. In triangle ABC, the bisectors of angles A and B meet at point M. The line through M which is parallel to BA intersects AC at X and BC at Y. If AX = 3 and BY = 5, compute the length of XY.

ANSWERS

1. 7
2. 60
3. 512
4. 31
5. 7
6. 8

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO SPRING, 1990

Part I: 10 Minutes

S90B7. Find the largest integer x such that the expression

$$\frac{24}{x+3}$$ represents an integer.

S90B8. What is the volume, in cubic units, of a cube whose surface area is 1200 square units?

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90B9. A seven-sided polygon has three sides of length $x^2 + y^2$, and four sides of length $x^2 - y^2$. Write an expression for the perimeter of this hexagon in terms of x and y.

S90B10. On a square sheet of 25 stamps, exactly two stamps will contain printing errors, and the printing errors occur at random places in the sheet. What is the probability that the two stamps with errors will be attached (by a common edge)?

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90B11. The first term of a geometric progression is 1 and the fifth term is 5. Find the product of the first five terms.

S90B12. Two circles have radii of 5 units and 9 units. Their centers are located 10 units apart. Find the length of their common chord.

ANSWERS

7. 21	9. $7x^2 - y^2$	11. $25\sqrt{5}$
8. $2000\sqrt{2}$	10. 2/15	12. $12\sqrt{14}/5$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER THREE SPRING, 1990

Part I: 10 Minutes

S90B13. Joe has \$1.24 in pennies, nickels, and quarters. He has the same number of each type of coin. How many coins does he have, in all?

S90B14. Two fractions are represented in base nine by 10/13 and 7/15. Reducing your answer to lowest terms, express their sum in base seven.

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90B15. In triangle PQR, angle R is a right angle. Compute the numerical value of the product

(sin P)(cos Q) + (sin Q)(cos P).

S90B16. In a circle, a 6-inch chord and a 14-inch chord are parallel and are 10 inches apart. The center of the circle is between the chords. Find the number of inches in the radius of the circle.

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90B17. If (a+b)/a = 6, and (b+c)/c = 9, compute the numerical value of a/c.

S90B18. The radii of circles O and P are 7 and 4 respectively, and OP = 8. A line is tangent to circle O at X and to circle P at Y. Compute the length of XY.

ANSWERS

13. 12	15. 1	17. 8/5
14. 5/4	16. $\sqrt{58}$	18. $\sqrt{55}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR SPRING, 1990

Part I: 10 Minutes

S90B19. A red dress sold originally for $50, and was marked down 10% for a sale. A green dress was marked <u>up</u> 10%, and sold for the same price as the sale price of the red dress. To the nearest dollar, what was the original price in dollars of the green dress?

S90B20. In a certain arithmetic progression, three times the third term is equal to six times the sixth term. Compute the numerical value of nine times the ninth term.

Part II: 10 Minutes NYCIML CONTEST FOUR SPRING, 1990

S90B21. The points A and B have polar coordinates $(4, 4\pi/3)$ and (6, 0) respectively. Compute the distance between the two points.

S90B22. A triangle has sides of lengths 13, 14, and 15. Compute the area of the triangle.

Part III: 10 Minutes NYCIML CONTEST FOUR SPRING, 1990

S90B23. If $\log_5 \sin x = -1/2$, compute the numerical value of $\cos^2 x$.

S90B24. If 4 men can paint 3 houses in 2 days, find the least positive integer x such that x men can paint y houses in 5 days, where y is also an integer.

ANSWERS

19. 41 or $41	21. $\sqrt{76}$ or $2\sqrt{19}$	23. 4/5
20. 0	22. 84	24. 8

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE SPRING, 1990

Part I: 10 Minutes

S90B25. Compute the numerical value of

$$\sqrt{5^5 + 5^5 + 5^5 + 5^5 + 5^5}$$

S90B26. On a rectangular coordinate system, the triangle with vertices at A(0,0), B(0,6) and C(10,0) is divided into two regions of equal area by the line whose equation is $y = k$. Compute k.

Part II: 10 Minutes NYCIML CONTEST FIVE SPRING, 1990

S90B27. If $f(x) = 1/(1-x)$ and $g(x) = 1 - (1/x)$, for $x > 1$, find the numerical value of $f(g(1990)) - g(f(1990))$.

S90B28. Three adjacent faces of a box (a rectangular parallelopiped) have areas 7, 14, and 18. Find the volume of the box.

Part III: 10 Minutes NYCIML CONTEST FIVE SPRING, 1990

S90B29. The lengths of the sides of a triangle are 2, 5, and x inches, and the area is x square inches. Compute the numerical value of x.

S90B30. A mathematician has two ten-inch candles A and B, which burn at equal and constant rates. Each candle will burn until the entire ten inches is consumed. She lights candle A at 11:40 AM, and lights candle B at 12:20 PM (forty minutes later). At 3:00 PM, she notices that the ratio of the length of A to B is half of what it was at 2:00 PM. When will candle A burn out?

ANSWERS

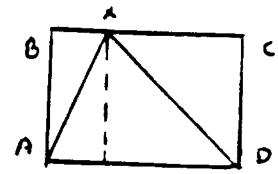
25. 125
26. $6 - 3\sqrt{2}$
27. 0
28. 42
29. $\sqrt{21}$
30. 3:20 or 3:20 PM

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION   CONTEST NUMBER ONE   SPRING, 1990

SOLUTIONS

S90B1. If a set has n elements, then it has 2^n subsets. Hence we must find two powers of 2 which differ by 96. By trial and error, we find that $2^7 - 2^5 = 128 - 32 = 96$, so set X has 7 elements.

S90B2. It is clear from the diagram that the area of triangle AXD is half that of the rectangle. Since the triangle is right-angled at X (try the converse of the Pythagorean theorem), its area is 5x12/2 = 30, and the area of the rectangle is 60.



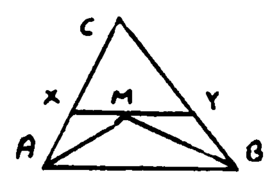
S90B3. We have $\sqrt{x\sqrt[3]{x}} = \sqrt{\sqrt[3]{x^4}} = ((x^4)^{1/3})^{1/2} = x^{4/6} = x^{2/3} = 64$, so $x = 64^{3/2} = 8^3 = 512$.

S90B4. If units of 1/24 inch were to be used alone, there would be 25 of them; namely, 0/24, 1/24, 2/24, . . . 24/24. A similar count shows that if units of 1/9 inch alone were to be used, there would be ten of them. Of these, 0/9, 3/9, 6/9, and 9/9 can be expressed as fractions with an integer numerator and a denominator of 24. Therefore, there are six "new" marks added for units of 1/9 inch, for a total of 31 marks.

S90B5. If the bunny removes six eggs, then it can have three pairs of eggs which are the same color, without getting three eggs which match. But if it removes one more egg, it must match one of the pairs.

This is an example of the "pigeon hole" principle: if you put n+1 objects into n boxes, then at least one box must hold more than one object.

S90B6. From the definition of angle bisector, and properties of parallel lines, m<XAM =m<MAB = m<AMX. Hence triangle AXM is isosceles, and AX = XM. Similarly, BY = YM. Therefore, XY = XM + MY = XA +YB = 8.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO SPRING, 1990

SOLUTIONS

S90B7. The number $x + 3$ must be a divisor of 24, and for x to be largest, we must choose the largest divisor. This is certainly the number 24 itself, which makes $x = 21$.

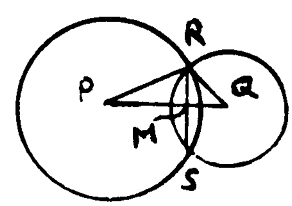
S90B8. A cube has six faces, so the area of each face is 200 square units. Thus an edge of the cube is $\sqrt{200} = 10\sqrt{2}$, and the volume is $(10\sqrt{2})^3 = 2000\sqrt{2}$ cubic units.

S90B9. The perimeter is $3(x^2 + y^2) + 4(x^2 - y^2) = 7x^2 - y^2$.

S90B10. There are $C(25,2) = 25x12$ choices for pairs of stamps altogether. Of these, there are 40 pairs which are attached: 4 in each row and 4 in each column. The probability of choosing one of these pairs is $40/25x12 = 2/15$.

S90B11. If the common ratio of the progression is r, then the progression is $1, r, r^2, r^3, r^4$, and the required product is r^{10}. Since $r^4 = 5$, $r^2 = \sqrt{5}$, and $r^{10} = (\sqrt{5})^5 = 25\sqrt{5}$.

S90B12. In the diagram, RM is half the common chord, and is the altitude of triangle PQR. We can compute its length by find the area of the triangle in two different ways.



First we use Hero's formula. The semiperimeter is 12, so the square of the area is (12)(3)(2)(7), and the area is $6\sqrt{14}$.

The area of triangle PRQ is also equal to (1/2)(10)RM, so $5RM = 6\sqrt{14}$, and $RS = 2RM = 12\sqrt{14}/5$.

Don't you wish the radii given in the problem were 6 and 8?

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER THREE SPRING, 1990

SOLUTIONS

S90B13. Joe must have four pennies, so he must have four of each coin, making 12 coins in all.

If Joe had 9 or more pennies, then he would have 9 or more quarters, and his sum would be too big.

S90B14. In base ten, the first fraction is represented by 9/12 = 3/4, and the second by 7/14 = 1/2. Their sum, in base ten notation, is 5/4. In base seven notation, this fraction is also represented by 5/4.

The answer $1\frac{1}{4}$ is also acceptable.

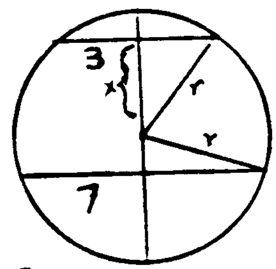
S90B15. The given expression is equal to sin (P + Q) = sin 90° (since the triangle is a right triangle), which is 1.

S90B16. With x and r as in the diagram,

$(10 - x)^2 + 49 = r^2$

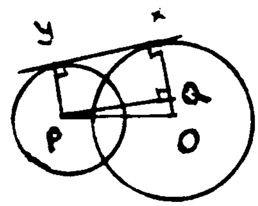
$x^2 + 9 = r^2.$

Subtracting gives x = 7, so $r^2 = 58$, and $r = \sqrt{58}$.



S90B17. We have (a+b)/a = 1 + b/a = 6, so b/a = 5. Similarly, 1 + b/c = 9, so b/c = 8. Therefore, a/c = = (a/b)(b/c) = (1/5)(8) = 8/5.

S90B18. In the diagram, PQ is drawn perpendicular to OX. Then QXYP is a rectangle, so XQ = PY = 4, and OQ = = XO-XQ = 3. In right triangle PQO, leg $QP^2 = 8^2 - 3^2 = 55$. Since XYPQ is a rectangle, XY = PQ = $\sqrt{55}$.

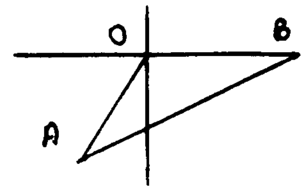


NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR SPRING, 1990

SOLUTIONS

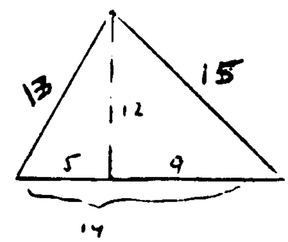
S90B19. We have $45 = 1.1x$, so $x = 45/1.1 = \$40.909$, or about \$41.

S90B20. If the first term is a and the common difference is d, then the third term is $a+2d$, the sixth term is $a+5d$, and setting these equal leads to $a = -8d$. The ninth term is $a+8d$, which is $-8d + 8d = 0$.



S90B21. In the diagram, angle AOB is $2\pi/3$. We use the law of cosines in triangle ABO:

$AB^2 = 4^2 + 6^2 - (2)(4)(6)(-1/2) = 76$, so $AB = \sqrt{76} = 2\sqrt{19}$.



S90B22. One could use Hero's formula, but it is easier to drawn in the altitude to the side of length 14. A glance at the diagram shows that this altitude has length 12, so the area of the triangle is $(1/2)(12)(14) = 84$.

S90B23. We have $\sin x = 1/\sqrt{5}$, so $\cos^2 x = 1 - \sin^2 x = 4/5$.

S90B24. One man can paint 3 houses in 8 days, so one man can paint 1 house in 8/3 days, and x men can paint y houses in $8y/3x = 5$ days. Thus $8y = 15x$. The smallest positive integer x for which y is also an integer is 8.

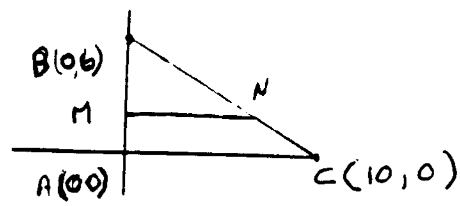
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION   CONTEST NUMBER FIVE   SPRING, 1990

SOLUTIONS

S90B25. We have:

$$\sqrt{5^5 + 5^5 + 5^5 + 5^5 + 5^5} = \sqrt{5 \cdot 5^5} = \sqrt{5^6} = 5^3 = 125.$$

S90B26. In the diagram, triangle BMN has half the area of triangle ABC, and the two triangles are similar. Since the ratio of the areas of two similar figures is the square of the ratio of their sides, we must have BM:BA = $1:\sqrt{2}$, so that BM = $6/\sqrt{2} = 3\sqrt{2}$. Hence AM = $6 - 3\sqrt{2}$, and this is also the value of k.



S90B27. $f(g(x)) = 1/(1 - (1 - 1/x)) = x$, and $g(f(x)) = 1 - (1-x) = x$. Hence $f(g(x)) - g(f(x)) = 0$ for any value of x.

The result can be "predicted" by noting that $g(x) = f(f(x))$.

S90B28. If the dimensions of the box are a, b, and c, we have ab = 7, bc = 14, ac = 18, so that $(abc)^2 = (7)(14)(18) = 7^2 \cdot 2^2 \cdot 3^2$, and the volume is abc = 42.

S90B29. The area of a triangle is half the product of two sides and the sine of the included angle. If the sides are of length 2 and x, and the angle is A, then $(1/2)2x \sin A = x$, which leads to $\sin A = 1$. This makes angle A a right angle, and the triangle is a right triangle with hypotenuse 5. The other leg of this triangle is $\sqrt{21}$.

S90B30. Candle B will burn out 2/3 hours after candle A, so we can measure the height of each by how much time it will take to burn down:

$$(1/2)(x+1)/(x + 5/3) = x/(x + 2/3), \text{ or } 3x^2 + 5x - 2 = 0,$$

and x = 1/3 or -2 (extraneous). Candle A will burn out at 3:20.

May 25, 1990

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1990 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S90S8	1
Senior B	S90B3	512 or -512 or both
Junior	S90J10	This question was eliminated from the competition. There is no such triangle.

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML