NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE SPRING, 1990

Part I: 10 Minutes

S90S1. If I give you three pennies, we will have an equal number of pennies. How many more pennies than you have do I have?

S90S2. A certain high school has more than 1000 students. Exactly one-sixth of these students own dogs. What is the smallest possible number of students in the high school?

Part II: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90S3. Points A and B are 12 units apart. Point Z is on line AB, and A is between B and Z. If AZ:ZB = 2:3, compute the length of AZ.

S90S4. The number 100x101x102x103 + 1 is a perfect square. What is its square root?

Part III: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90S5. A teacher does one day of work grading the Regents examinations. She is joined by a second teacher on the second day. These two teachers are joined by a third on the third day, and they finish grading the Regents at the end of the third day. If all teachers grade at the same rate, how long would it have taken all three teachers to grade the Regents, if they had worked together all the time?

S90S6. Find the smallest pair of consecutive integers, each of which is divisible by the cube of a natural number greater than one.

ANSWERS

1. 6	3. 24	5. 2
2. 1002	4. 10301	6. 80, 81 (in any order)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO SPRING, 1990

Part I: 10 Minutes

S90S7. In quadrilateral MNLQ, angles L and N are right angles, and the tangent of angle M is 2/3. If LQ = x, MN = 2x, and LN = x-2, compute the length of diagonal QN.

S90 S8. Find the smallest perfect square which leaves a remainder of 1 upon division by 19.

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90S9. In a certain restaurant , a mixture of 1 part water to two parts wine (by volume) is sold as "Chablee". In the same restaurant, another mixture of 1 part water to 4 parts wine (by volume) is sold as "Zinfoundel". If 9 gallons of "Chablee" is mixed with a certain amount of "Zinfoundel", an new mixture is created which consists of 1 part water to 3 parts wine, by volume. How many gallons of the new wine is created?

S90S10. None of the numbers x, y, or z is equal to 0, and $x + y + z = 2$. If $1/x + 1/y + 1/z = 0$, compute the numerical value of $x^2 + y^2 + z^2$.

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90S11. Point P is a vertex of a regular decagon (regular 10-gon) which is inscribed in a circle of radius 5. Nine line segments connect P with the other nine vertices of the decagon. Find the sum of the squares of these nine line segments.

S90S12. Suppose that $\log_{10}2 = .3$. On the basis of this approximation, compute the number of digits in the decimal numeral representing 2^{100}.

ANSWERS

7. $2\sqrt{13}$

8. 324

9. 24 or 24 gallons

10. 4

11. 500

12. 31

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE SPRING, 1990

Part I: 10 Minutes

S90S13. Find all real values of x such that

$$\sqrt{6 - 4x - x^2} = x + 4 .$$

S90S14. Three friend are in a class of 30. The teacher is about to choose ten students from the class, at random, to form a committee. What is the probability that all three friends will be chosen?

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90S15. For certain natural numbers n, the quantity $n^2 - 8n + 4$ is a multiple of 13. Find the sum of the two smallest such natural numbers.

S90S16. Inside regular hexagon ABCDEF, and on the same plane, squares BCXY and FEWV are constructed. If AB = 1, find the area of the region in which these two squares overlap.

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90S17. In triangle ABC, the degree-measures of angles B and C are 60 and 39, respectively. Points X and Y are on side BC, with X nearer to B, such that $m\angle BAX = m\angle XAY = m\angle YAC$. If AT is the altitude to side BC, find the degree-measure of $\angle XAT$.

S90S18. In a sequence of natural numbers, each number after the first number is one more than the sum of all the previous numbers. If the tenth number is 1280, find the first number.

ANSWERS

13. -1	15. 21	17. 3 or 3°
14. 6/203	16. $2 - \sqrt{3}$	18. 4

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR SPRING, 1990

Part I: 10 Minutes

S90S19. How many natural numbers divide 240 (without remainder)? Remember that 1 and 240 must be counted among these.

S90S20. Square ABCD is cut out of cardboard with AB = 1. Isosceles right triangles are cut off the four corners of the square, leaving a regular octagon. Find the area of this octagon.

Part II: 10 Minutes NYCIML CONTEST FOUR SPRING, 1990

S90S21. For <u>how many</u> real values of x is $0 \leq x \leq 2\pi$ and $2 \sin x + 3 \sin 2x = 0$?

S90S22. The integer 1000 is divided by a perfect square which is less than 500. What is the largest possible remainder?

Part III: 10 Minutes NYCIML CONTEST FOUR SPRING, 1990

S90S23. In triangle ABC, $m\angle A = 75^\circ$, $m\angle C = 45^\circ$, and AC = 20. A circle passing through point A is tangent to line BC at T. If $AT \perp BC$, compute the length of the radius of this circle.

S90S24. Find the smallest positive (non-zero) value of x such that

$$\sin 7x + \sin 9x + \sin 11x + \sin 13x = 0.$$

ANSWERS

19. 20	21. 5	23. $5\sqrt{2}$
20. $2\sqrt{2} - 2$	22. 278	24. $\pi/10$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION   CONTEST NUMBER FIVE   SPRING, 1990

Part I: 10 Minutes

S90S25. Circles O and P (with centers O and P respectively) are tangent <u>internally</u> at point T. Point X is any point on circle P. If OT = 4 and PT = 10, find the length of the largest possible tangent segment from X to circle O.

S90S26. The tremendous number $2^7 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11^3$ is divisible by many perfect squares. By how many?

Part II: 10 Minutes   NYCIML CONTEST FIVE   SPRING, 1990

S90S27. If x and y are both real numbers, find the smallest possible value of the expression

$$x^2 + 3y^2 - 10x + 12y + 30.$$

S90S28. A committee of 15 people must have a meeting. Ten of these people live in New York City, and five live in Albany. The straight road between these two cities is 150 miles long, and the committee can arrange to meet anywhere along this road. If travel expenses are ten cents per person per mile, what is the smallest number of <u>dollars</u> which must be laid out for the committee's travel expenses for a <u>one way</u> trip to the meeting?

Part III: 10 Minutes   NYCIML CONTEST FIVE   SPRING, 1990

S90S29. Find the smallest positive integer x such that the value of the expression $x^3 - 6x^2 + 11x - 6$ is <u>not</u> zero.

S90S30. Points A, B, C, and D are consecutive vertices of a regular polygon, and 1/AB = 1/AC + 1/AD. How many vertices does the polygon have?

ANSWERS

25. $\sqrt{240}$ or $4\sqrt{15}$	27. -7	29. 4
26. 48	28. 75 or \$75	30. 7

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE SPRING, 1990

SOLUTIONS

S90S1. (Mine) - 3 = (Yours) + 3, so (Mine) = (Yours) +6.

S90S2. We are seeking the smallest multiple of six that is greater than 1000, which is 1002.

S90S3. If AZ = x, then (see diagram) ZB:AZ = 3:2 = = (x+12)/x, or 1 + 12/x = 3/2, so that 12/x = 1/2, and x = 24.

S90S4. If N is the square root, then 100x101x102x103 = = N^2-1 = (N+1)(N-1) = 100x103 x 101x102 = = (100^2 + 300)(100+1)(100+2). These numbers differ by 2, so they must be identical with the algebraic factors. Hence N - 1 = 100^2 + 300 = 10300, N = 10301.

In general, one more than the product of four consecutive integers will be a perfect square.

S90S5. It took it took 6 teacher-days to grade the Regents, so it would have taken the three of them two days working together.

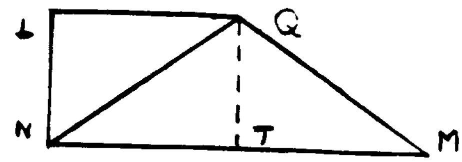
S90S6. In the interests of finding a "small" pair (and this expression is not yet well-defined in this problem), we may guess that the cubes are 8 and 27. We must then solve the equation 8a - 27b = $\pm$ 1 in integers.

Standard techniques for this chore are given in any book on elementary number theory. In this case, we can note that b must be odd, and trial and error shows that b = 3 works. The numbers are 80 and 81 (in any order).

Now the next largest perfect cube is 64, and this leads to no solution unless it is at least doubled. Hence each of the numbers we have found is smaller than any member of the "next largest" pair, which is what was meant by the "smallest pair".

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO SPRING, 1990

SOLUTIONS



S90S7. From right triangle QTM (see diagram), and rectangle QLTN, we see that NT = QL = x, so MT = MN-TN = 2x-x = x. Since tan M = QT/MT = 2/3, QT = 2x/3 = LN = x-2. This equation leads to x = 6. Then, from right triangle LNQ, $QN^2 = 4^2 + 6^2 = 52$, so $QN = \sqrt{52} = 2\sqrt{13}$.

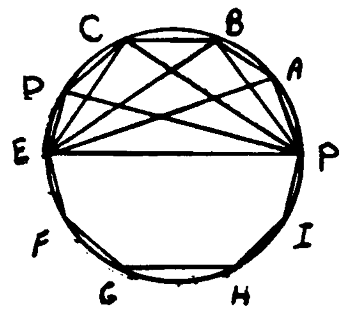
S90S8. We need the smallest N such that $N^2 = 19k+1$, or $N^2 - 1 = 19k = (N+1)(N-1)$, so N = 20 or 18. For the smallest solution, we take N = 18, and the square is 324.

S90S9. If we mix a gallons of "Chablee" with b gallons of "Zinfoundel", then (equating the amount of water in each of the three mixtures), $a/3 + b/5 = (a+b)/4$, or $20a + 12b = 15a + 15b$, or $5a = 3b$, and the ratio a:b is 3:5. Hence 9 gallons of "Chablee" require 15 gallons of "Zinfoundel", to make 24 gallons of the new "wine".

S90S10. By direct computation,

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy+yz+zy),$$
$$1/x + 1/y + 1/z = (xy+yz+xz)/xyz = 0.$$

Hence $xy+yz+xy = 0$, and $x^2 + y^2 + z^2 = (x+y+z)^2 = 4$.



S90S11. In the diagram, PC = BE and AE = PD (by symmetry), so $PA^2+PB^2+PC^2+PD^2 = PA^2+PB^2+BE^2+AE^2 = 2PE^2 = 200$ (since triangles PCE, PBE are right-angled at C and B respectively). Similarly, $PF^2+PG^2+PH^2+PI^2 = 2PE^2 = 200$ as well. Adding to these totals the value of $PE^2 = 100$, we find that the required sum of squares is 500.

S90S12. If there are N digits in the required numeral, then $10^{N-1} \leq 2^{100} < 10^N$, or $N-1 \leq 100 \log_{10}2 < N$, or $N-1 \leq 30 < N$, and N = 31.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE SPRING, 1990

SOLUTIONS

S90S13. Squaring both sides gives $6 - 4x - x^2 = x^2 + 8x + 16$, or $2x^2 + 12x + 10 = 0$. This quadratic leads to $x = -1, -5$. However, the root -5 is extraneous to the original problem.

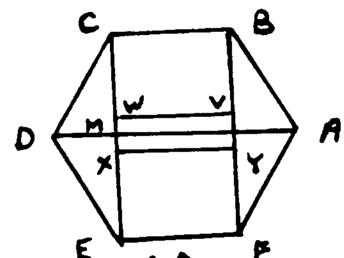
Do not give credit for more than one value in the answer.

S90S14. The required probability is $\binom{27}{7}$ divided by $\binom{30}{10}$,

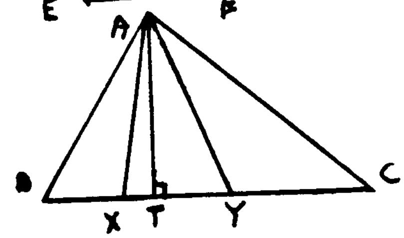
or $\dfrac{27 \times 26 \times 25 \times 24 \times 23 \times 22 \times 21}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} \times \dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23 \times 22 \times 21}$
$= 6/203$.

S90S15. The sum of two multiples of 13 is again a multiple of 13. Hence if $n^2 - 8n + 4$ is a multiple of 13, then so is $n^2 - 8n + 4 + 13n = n^2 + 5n + 4 = (n+4)(n+1)$. Since n is a natural number, this product is positive. To get the smallest values of n, we take the value of the product to be 13. But then we must have $n+4 = 13$ or $n+1 = 13$, which leads to $n = 9$, $n = 12$. The sum of these two numbers is 21.

S90S16. The region is a rectangle whose width is clearly 1 (see diagram). To find the height, we draw CE. From 30-60-90 triangle CDM, or otherwise, $CM = \sqrt{3}/2$, so $CE = \sqrt{3}$. But $CX + WE = 2 = CE + WX$, so $WX = 2 - \sqrt{3}$.



S90S17. In right triangle BAT, $m\angle BAT = 90-60=30$. Since $m\angle BAC = 81$, $m\angle BAX = 27$, and $m\angle XAT = m\angle BAT - m\angle BAX = 30 - 27 = 3$.



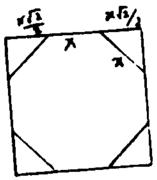
S90S18. If a is the first term, the sequence is a, a+1, 2a+2, 4a+4, ... and the n^{th} term is $2^{n-2}(a+1)$. Here, $2^8(a+1) = 1280$, or $256(a+1) = 1280$, $a+1 = 5$, and $a = 4$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR SPRING, 1990

SOLUTIONS

S90S19. Since $240 = 2^4 \cdot 3 \cdot 5$, there are five choices for the number of factors of 2 a divisor can have, two choices for the number of factors of 3, and two choices for the number of factors of 5. Altogether, there are 5x2x2 = 20 possible divisors of 240. This count includes 1 and 240 among the divisors.

S90S20. If a side of the resulting octagon is x, then (see diagram) $x\sqrt{2}/2 + x + x\sqrt{2}/2 = 1$, which leads to $x = 1/(1+\sqrt{2}) = \sqrt{2} - 1$. The cut off triangles can be assembled into a square with side x. Therefore, the area of the octagon is $1 - x^2 = 1 - (\sqrt{2}-1)^2 = 1-(3-2\sqrt{2}) = 2\sqrt{2} - 2$.



Accept any mathematically equivalent answer.

S90S21. We have 2 sin x + 6 sin x cos x = 0, or
2 sin x (1 + 3 cos x) = 0.

If sin x = 0, then x = 0, π, or 2π.

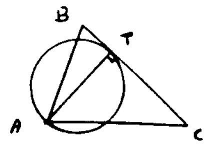
If 1 + 3 cos x = 0, then cos x = -1/3. A quick sketch of the graph of the function y = cos x will show two more values of x for which this is true, for a total of five values altogether.

S90S22. This difficult trial-by-error can be made more palatable by starting from the top and work down, since larger squares are more likely to leave large remainders.

1000 = 484x2 + 32
= 441x2 + 118
= 400x2 + 200
= 361x2 + 278
= 324x3 + 28
= 289x3 + 133, and the remaining squares are less than 278, which is the largest remainder so far. Since no remainder can exceed the corresponding divisor, the largest remainder must be 278.

But how can you address this problem more generally?

S90S23. From right triangle ATC, m<TAC = 45, so AT = $10\sqrt{2}$. But a line perpendicular to a tangent to a circle at its point of contact passes through the center of the circle. Therefore AT is a diameter of the circle, and its radius is $5\sqrt{2}$.



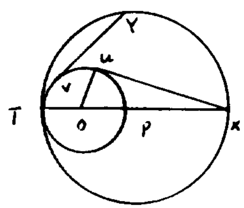
S90S24. This is an exercise in sums-to-products formulas. If we can express the given sum as a product, then we can solve by setting each factor equal to zero. To this end, we have:
(sin 7x + sin 13x) + (sin 9x + sin 11x)
= 2 sin 10x cos 3x + 2 sin 10x cos x
= 2 sin 10x (cos 3x + cos x)
= 2 sin 10x (2 cos 2x cos x)
= 4 sin 10x cos 2x cos x
= 0,

so sin 10x = 0 OR cos 2x = 0 OR cos x = 0. By examining each case in turn, it is not hard to see that the smallest value will occur when 10x = π, so x = π/10.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE SPRING, 1990

SOLUTIONS

S90S25. It is not difficult to guess that point X must be diametrically opposite T. In this position, $XU^2 = XO^2 - UO^2 = (XT - OT)^2 - UO^2 = 16^2 - 4^2 = 240$. Hence $XU = \sqrt{240} = 4\sqrt{15}$.



If you are not sure that this position of X results in the longest tangent, and you think that (say) YV is longer, note that $YV^2 = YO^2 - OV^2$, and we can show that YV < XU by showing that YO < XO. But this last inequality follows from the fact that XO = XP+PO = YP+PO > YO in triangle OPY.

S90S26. If a prime p divides a perfect square, than p^2 must also divide the perfect squre. Hence the primes enter into the factorization of a perfect square with even exponents. Representing the given number as $2 \cdot 2^6 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 11^2$ shows that a divisor can have 0, 2, 4, or 6 factors of 2,
0, 2, or 4 factors of 3,
0 or 2 factors of 7,
0 or 2 factors of 11,
for a total of 4x3x2x2 = 48 perfect square divisors in all.

S90S27. We complete the square twice. The given expression is equal to:
$(x^2 - 10x + 25) + 3(y^2 + 4y + 4) - 25 - 12 + 30 =$
$= (x-5)^2 + 3(y+2)^2 - 7$. This expression is smallest when x-5 = y+2 = 0 (since a square cannot be negative), and the minimal value is -7.

S90S28. Suppose the committee agrees to meet x miles from New York. Then the travel expenses (in cents) will be $10 \cdot 10x + 5 \cdot 10(150-x) = 100x + 7500-50x = 50x + 7500$. This is clearly minimal if x = 0; that is, if the meeting is arranged in New York. The cost will then be 7500 cents, or 75 dollars.

This problem was based on an article in the excellent Russian journal KBAHT, Jan 87, p. 37 by Gutenmacher and Rabbot.

S90S29. By trial and error, the polynomial is zero for x = 1, 2, 3. Since it is a cubic polynomial, it cannot be zero for any other value of x. Hence the smallest positive integer for which the polynomial is not zero is 4.

S90S30. We can consider the polygon as having been inscribed in a circle of <u>diameter</u> 1. In such a circle, the length of a chord subtended by an inscribed angle whose measure is x is sin x (figure i). If angle BAC = x, then (figure ii), AB = sin x, AC = sin 2x, AD = sin 3x, and we have:

1/ sin x = 1/sin 2x + 1/sin 3x , or

sin 2x sin 3x = sin x sin 2x + sin x sin 3x. We now use sums-to-products and products-to-sums, to reduce this equation to a product equalling zero. The problem is solved by setting each factor to zero (compare problem S90S24):

2 sin 2x sin 3x = 2 sin x sin 2x + 2 sin x sin 3x
cos (2x-3x) - cos (2x+3x) = cos (x-2x) - cos (x+2x) + cos (x-3x)-cos (x+3x)
cos x - cos 5x = cos x - cos 3x + cos 2x - cos 4x
(using the fact that cos A = cos (-A);
-cos 2x + cos 3x + cos 4x - cos 5x = 0

-(cos 2x + cos 5x) + (cos 4x + cos 3x) = 0
-2 cos (7x/2) cos (3x/2) + 2 cos (7x/2) cos (x/2) = 0
cos(7x/2)[cos 3x/2 + cos x/2] = 0
cos(7x/2)[2 cos 2x/2 cos x/2] = 0.

Setting each factor in turn equal to zero, we find that the last two factors do not give a solution (angle x must be acute). The first factor, however, gives $7x/2 = \pi/2$, so $x = \pi/7$, and the polygon has seven sides.

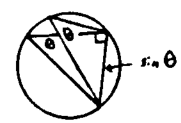


Fig i

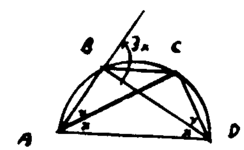


Fig ii

May 25, 1990

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1990 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S90S8	1
Senior B	S90B3	512 or -512 or both
Junior	S90J10	This question was eliminated from the competition. There is no such triangle.

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML