NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE SPRING, 1990

Part I: 10 Minutes

S90J1. The average (arithmetic mean) of x and 4x is 20. Compute x.

S90J2. Find the units digit of the decimal numeral representing the number $11^{11} + 14^{14} + 16^{16}$.

Part II: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90J3. Find a two digit decimal numeral that represents a number which is twice the product of its digits.

S90J4. The set S consists of all four-digit decimal numbers which contain only the digits 2 and 4. Compute the sum of all the numbers in S.

Part III: 10 Minutes NYCIML CONTEST ONE SPRING, 1990

S90J5. The total surface area of a cube, expressed in square centimeters, is equal to the volume of the cube, expressed in cubic centimeters. Compute the length, in centimeters, of a side of the square.

S90J6. In triangle ABC, AB = 6, BC = 7, and AC = 8. AH, BK and CL are altitudes to sides BC, AC, and AB respectively. Compute the perimeter of triangle HKL.

ANSWERS

1. 8	3. 36	5. 6 or 6 cm.
2. 3	4. 53328	6. 315/32 or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER TWO SPRING, 1990

Part I: 10 Minutes

S90J7. Carol is collecting wild berries. She finds that the berries lose 10% of their weight by drying out on the way home. How many pounds of berries must Carol collect, if she wants to arrive home with 100 pounds of berries?

S90J8. A rectangle has sides 4 and 8. A parallelogram has the same area as the rectangle, and each side is equal to one of the diagonals of the rectangle. Find the sine of the acute angle between the sides of the parallelogram.

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90J9. The average (arithmetic mean) of a and b is 10, while the average of b and 10 is c/2. What is the average of a and c?

S90J10. In right triangle ABC, leg BC = 3 and leg AC = 4. Point P is between A and C, and point Q is between B and C. The line PQ divides the area of the triangle into two regions of equal area. It also divides the perimeter of the triangle into two paths of equal length. If PC > QC, compute the length of PC.

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1990

S90J11. Find all real numbers x such that

$$4x - 13\sqrt{x} + 10 = 0.$$

S90J12. Square ABCD is the base of a pyramid, point P is its vertex, and PA = PB = PC = PD = AB. Point X is the intersection of diagonals AC and BD of the square. A plane cuts PA, PC, and PX at points R, S and T respectively. If PR = 6 and PS = 8, compute the length of PT.

ANSWERS

7. 111 1/9 or 1000/9 or equivalent.
9. 15
11. 25/16, 4: both required, in any order
8. 2/5
10. $3 + \sqrt{3}$
12. $24\sqrt{2}/7$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER THREE SPRING, 1990

Part I: 10 Minutes

S90J13. Two adjacent sides of a square are represented by x and 4x - 12. Find the area of the square.

S90J14. The Oracle of Flatbush tells the truth whenever it chooses to answer a question, and the probability that it will choose to answer a given question is 1/3. A student plans to ask the Oracle three times for his grade on the Course III Regents. What is the probability that she will get the answer?

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90J15. $\dfrac{(q)\cdot(q)\cdot(q)}{(q + q + q)} = 48$, and $q > 0$, find q.

S90J16. Transylvanian license plates consist of exactly three letters. Two license plates are considered identical if (and only if) they contain the same three letters in the same order. How many Transylvanian license plates are possible, if the letter Q must be followed directly by the letter U?

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1990

S90J17. A commercial lease calls for a rental of $1000 the first month, and an increase of ten percent each month over the previous month's rate. How much will it cost to lease this property for six months?

S90J18. In pyramid PWXYZ, WXYZ is a rectangle, and PW = PX = PY = PZ. A plane cuts segments PW, PX, PY, PZ at points A, B, C, and D respectively. If PA = 2, PB = 12, PC = 4, compute the length of PD.

ANSWERS

13. 16 or
16 square units.

14. 19/27

15. 12.

16. 15675

17. $7715.61

18. 3/2

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE SPRING, 1990

SOLUTIONS

S90J1. We have x + 4x = 40, or 5x = 40, so x = 8.

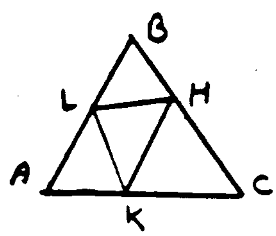
S90J2. By doing the multiplications, and keeping track only of the units digit, it is not difficult to see that a power of 11 must end in the digit 1, a power of 16 must end in the digit 6, and an even power of 14 must end in the digit 6 (odd powers end in the digit 4). Hence the units digit of the required number is obtained from the sum 1 + 6 + 6 = 13, and is equal to 3.

S90J3. If the number is 10t + u, then 2tu = 10t + u, or u(2t - 1) = 10t. It follows that either u or 2t-1 is a multiple of 5, and since u and t are digits, one of them must equal 5. If 2t-1 = 5, then t=3, and 6u = 30+u, so u = 6. The numeral 36 satisfies the conditions of the problem. Further investigation will show that this solution is unique.

S90J4. To form a number in S, we can choose the first digit in two ways (a 2 or a 4), the second digit in two ways, and so on. This count gives a total of sixteen numbers in S. If we write them in a column, to add in the usual fashion, then the units column will contain 8 2's and 8 4's, so its sum will be 48. Hence the total will have a units digit of 8. The tens column sums in the same way, with a "carryover" of 4 from the units column, for a sum of 52. Similarly, the hundreds column (with carryover) sums to 53, and the thousands column to 53. The sum is therefore 53328.

S90J5. If a side of the cube is s, then $s^3 = 6s^2$, so s = 6.

S90J6. From right triangles ABK, ALC, we find that AK = AB cos A, AL = AC cos A. This means that two sides of triangles ALK, ABC are in proportion, and the included angles (angle BAC) are equal. Hence the triangles are similar, and KL = BC cos A.



In the same way, we find that HL = AC cos B and HK = AB cos C. We can now use the law of cosines in triangle ABC to compute these lengths:

cos A = 17/32; KL = 8x(17/32) = 119/32
cos B = 1/4; HL = 8x(1/4) = 2
cos C = 11/16; HK = 6x(11/160 = 33/8

The perimeter of the triangle is the sum of these numbers, which is 315/32.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER TWO SPRING, 1990

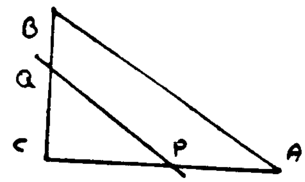
SOLUTIONS

S90J7. We have $.9x = 100$, so $x = 100/.9 = 1000/9 = 111\ 1/9$.

S90J8. The area of the rectangle is 32, and its diagonals are both equal to $\sqrt{80}$ (using the Pythagorean theorem). If the required angle is A, then $(\sqrt{80})(\sqrt{80}) \sin A = 32$, so $\sin A = 32/80 = 2/5$.

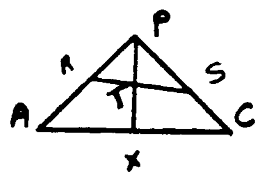
S90J9. We have $a + b = 20$, $b + 10 = c$. Hence $a + c = a + b + 10 = 20 + 10 = 30$, and $a + c = 30$. This means that the average of a and c is 15.

S90J10. The area of the given triangle is 6, and its perimeter is 12. If $CP = a$, $CQ = b$ (see diagram), then $a + b = 6$, $ab = 6$. Hence a and b are the roots of the quadratic equation $x^2 - 6x + 6 = 0$. Solving this equation gives $x = 3 \pm\sqrt{3}$. Either value can be taken for a, and the other value for b. The problem calls for the larger value of a, which is $3 + \sqrt{3}$.



S90J11. Let $y = \sqrt{x}$. Then $x = y^2$, and we can rewrite the given equation as a quadratic in y: $4y^2 - 13y + 10 = (4y - 5)(y-2) = 0$, so $y = 5/4, 2$. This gives $x = 25/16, 4$. Note that the statement of the problem precludes negative values for x.

S90J12. It is not difficult to see that triangle PAC is right-angled at P (it is, in fact, an isosceles right triangle). The diagram shows this triangle, in which PX is an angle bisector. The Pythagorean theorem gives $RS = 10$. We can find PT by computing the area of triangle PRS in two different ways.



Using absolute value for area, we have $|PRS| = (1/2)(6)(8) = 24 = |PRT| + |PST| = (1/2)(6x)(\sin 45^\circ) + (1/2)(8x)(\sin 45^\circ) = (\sqrt{2}/4)(14x)$, so that $x = 24\sqrt{2}/7$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER THREE SPRING, 1990

SOLUTIONS

S90J13. Since $x = 4x - 12$, $3x = 12$, and $x = 4$. The area is 16 square units.

S90J14. The probability that she will not get the answer on any one try is 2/3. Therefore, the probability that she will not get the answer at all in her three tries is $(2/3)^3 = 8/27$. The probability that she will get the answer on one of her tries is $1 - 8/27 = 19/27$.

If the student gets an answer the first time, she may or may not choose to ask again (she will either get the same information, or no information at all). This circumstance does not alter the solution to the problem.

S90J15. We have $q^3/3q = q^2/3 = 48$, so $q^2 = 144$, and $q = 12$.

S90J16. If the letter Q were not in the alphabet, there would be 25x25x25 = 15625 possible license plates. We must add to this the number of plates which include the letter Q. If Q is the first letter, then the second letter is U, and any letter other than Q can be third. This makes 25 new plates. If Q is the second letter, then the third letter must be U, and the first letter can be any letter other than Q. This gives 25 more possible plates, for a total of 15675.

There are no plates with the letter Q last, or with more than one letter Q.

S90J17. The first month's rental is $1000
The second month's rental is $1000 + $100 = $1100
The third month's rental is $1100 + $110 = $1210.
The fourth month's rental is $1210 + 121 = $1331.
The fifth month's rental is $1331 + $133.10 = $1464.10
The sixth month's rental is $1464.10 + $146.41 = $1610.51.

The total rental is $7715.61. Note that the compounding over six months produces an increase of more than 69% over the original month's rate.

S90J18. We let PA = a, PB = b, PC = c, PD = d, and solve the problem in general.

The diagram shows a section along the plane through P, A, and C (compare problem S90J12). Here, triangle WPY is not right-angled, but it is isosceles, so that the altitude of the pyramid makes equal angles with PW, PX, PY, and PZ. Let us call that angle x. PQ is that portion of the altitude of the pyramid which lies inside triangle PAC, so it is an angle bisector of the triangle. We will get an expression for PQ in terms of a and c, then a similar expression for PQ in terms of b and d. A solution to the problem will come from the equality of these two expressions.

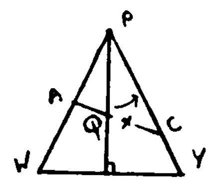
We get the first expression for PQ by computing the area of triangle PAC (just as in problem S90J12). Using absolute value for area, and letting PQ = q, we have:

$|PAC| = (1/2)ac \sin 2x = |PAQ| + |QPC| = (1/2)aq \sin x + (1/2)cq \sin x$. Using the usual identity for sin 2x, this gives:

$2ac \sin x \cos x = aq \sin x + cq \sin x$, or $2ac \cos x = aq+cq$, or $q = (2ac \cos x)/(a+c)$.

If we cut the pyramid along the plane of PBD, we can do the same calculations, to get that $q = (2bd \cos x)/(b+d)$. Equating these two (and noting that the angle x is the same in both expressions), gives $bd/(b+d) = ac/(a+c)$. Inverting each side, we can write this as $1/a + 1/c = 1/b + 1/d$.

In this problem, a = 2, b = 12, c = 4, so $1/2 + 1/4 = 1/12 + 1/d$, and $d = 3/2$.



May 25, 1990

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1990 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S90S8	1
Senior B	S90B3	512 or -512 or both
Junior	S90J10	This question was eliminated from the competition. There is no such triangle.

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML