NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE FALL, 1989

Part I: 10 Minutes

F89B1. Find the smallest integer which is greater than 1000 and which is divisible by 5 and by 13, but not by 4.

F89B2. Two sides of a triangle measure 6 and 11 centimeters. If the length, in centimeters, of the third side is also an integer, how many possible lengths can this third side have?

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Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89B3. If $\log_7 x = .47$, compute $\log_{49} x^2$.

F89B4. If r and s are the roots of the equation $x^2 + x + 12345 = 0$, compute the numerical value of $1/r + 1/s$.

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Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89B5. Compute the numerical value of

$$\sin(\text{Arcsin } 3/5 + \text{Arccos } 7/25),$$

where Arcsin and Arccos denote principal value.

F89B6. The square of the complex number a + bi is −9 + 40i. If a and b are real numbers, i represents the imaginary unit, and a > 0, compute the numerical value of a + b.

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ANSWERS

| | | |
|---|---|---|
| 1. 1105. | 3. .47 | 5. 117/125 |
| 2. 11 | 4. −1/12345. | 6. 9 |

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO FALL, 1989

Part I: 10 Minutes

F89B7. How many diagonals does a convex polygon with 20 sides have?

F89B8. Find the smallest positive real number x such that $0 < x < \pi$ and $\sin x = \cos 2x$.

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Part II: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89B9. Leaving your result in terms of $\pi$, find the number of degrees in the measure of an angle whose radian measure is 1.

F89B10. If r and s are the roots of the equation

$3x^2 - 7x + 1 = 0$, compute the value of $r^3s + rs^3$.

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Part III: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89B11. What is the radius of a circle in which a chord 10 units long is 5 units from the center?

F89B12. At x minutes past noon, where $x < 60$, the hands of a clock make an angle of 60 degrees with each other. Find the two possible values of x.

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**ANSWERS**

7. 170
8. $\pi/6$
9. $180/\pi$
10. 43/27
11. $5\sqrt{2}$ or equivalent
12. 120/11 and 600/11
See note in solutions

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION     CONTEST NUMBER THREE  FALL, 1989

Part I: 10 Minutes

F89B13. A convex polygon has 119 diagonals. How many sides does it have?

F89B14. The first term of an arithmetic progression of four terms is 5, and the last term is 2117. Find the sum of the middle two terms.

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Part II: 10 Minutes     NYCIML CONTEST THREE  FALL, 1989

F89B15. In trapezoid ABCD, bases AB and CD are parallel. Point E is between A and D, and point F is between B and C, such that AE:ED = BF:FC = 3:2. If AB = 7 and CD = 10, compute the length of EF.

F89B16. What is the remainder when $10^{1989}$ is divided by 7?

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Part III: 10 Minutes     NYCIML CONTEST THREE  FALL, 1989

F89B17. Find the sum of all the integers between 100 and 1000 which are multiples of 7.

F89B18. The line $y = mx + b$ is tangent to the parabola $y = x^2$, and passes through the point whose coordinates are (1,-5). Find the largest possible value of m.

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ANSWERS

13. 17     15. 44/5     17. 70336

14. 2122     16. 6     18. $2 + 2\sqrt{6}$ (or equivalent)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION    CONTEST NUMBER FOUR   FALL, 1989

Part I: 10 Minutes

F89B19. If the cost of nine pieces of candy were increased (from what it is now) to 33 cents, then the cost of a dozen pieces of candy would rise by 5 cents. What is the actual cost, in cents, of a dozen pieces of candy?

F89B20. How many non-congruent right triangles are there whose sides all have integral lengths, and with one leg of length 40?

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Part II: 10 Minutes    NYCIML CONTEST FOUR   FALL, 1989

F89B21. Susanna chose at random an integer between 5 and 15, and evaluated the polynomial $x^2 - 29$ for the integer she chose. What is the probability that her result is a prime number?

F89B22. The square of the number $\sqrt{a} + \sqrt{b}$ , where a and b are positive integers and $a < b$, is $5 + 2\sqrt{6}$. Find the ordered pair (a,b) of positive integers.

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Part III: 10 Minutes    NYCIML CONTEST FOUR   FALL, 1989

F89B23. A rectangular garden has dimensions 20 meters by 30 meters. It is surrounded by a border hedge which is one meter wide, forming a larger rectangle all together. Compute the total area of the border, in square meters.

F89B24. Three numbers form a geometric progression. Their sum is 21, and the sum of their reciprocals is 7/12. Find all possible values for the common ratio of the geometric progression.

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ANSWERS

19. 39 or 39 cents    21. 1/3    23. 104 or 104 $meters^2$

20. 7    22. (2,3)    24. 2, 1/2, $(-9 \pm \sqrt{65})/4$
see note in solution

# NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
## SENIOR B DIVISION CONTEST NUMBER FIVE FALL, 1989

Part I: 10 Minutes

F89B25. Two sides of a triangle measure 10 and 12 centimeters, and the altitude to the shorter of the two sides measures 9 centimeters. Find the measure, in centimeters, of the altitude to the longer of the two sides.

F89B26. A rectangular picture has a frame which is one inch wide. Both together form a larger rectangle. The total area of the frame alone is 100 square inches. Compute all possible values of the perimeter of the picture without the frame.

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Part II: 10 Minutes NYCIML CONTEST FIVE FALL, 1989

F89B27. Find the ordered triple $(x, y, z)$ of real numbers such that $x + y = 5$, $y + z = -7$, and $x + z = 12$.

F89B28. The numbers $p, q, r, s, t$ form a geometric progression (in that order). If $\log_{77} p + \log_{77} t = 44$, find the numerical value of $\log_{77} r$.

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Part III: 10 Minutes NYCIML CONTEST FIVE FALL, 1989

F89B29. Two urns contain a number of identical marbles. The first contains 12 black marbles, while the second contains 6 black marbles and six white marbles. A marble is chosen at random from the second urn, and placed in the first urn. A marble is then chosen from the first urn. What is the probability that the marble chosen from the first urn is black?

F89B30. If $a \neq 0$ and if $x+y=a$ and $x^3+y^3 = b$, write an equation expressing $x^2 +y^2$ explicitly in terms of a and b.

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ANSWERS

25. 15/2 or 15/2 cm. or equivalent
26. 96
27. (12, -7, 0) ordered triple required.
28. 22
29. 25/26
30. $x^2+y^2=(a^3+2b)/3a$ equation required

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE FALL, 1989

SOLUTIONS

F89B1. An integer which is divisible by 5 and by 13 is divisible by 65. Since 1000/65 > 15, the smallest such integer greater than 1000 is 65x16 = 1040. But this number is a multiple of 4. The next multiple of 65, however, is not. This number is 1105.

F89B2. The length of a side of a triangle must be less than the sum of the lengths of the other two sides. If the length of the third side is N, then N must be less than 17. By the same restriction, 11 must be less than N + 6, so that N must be greater than 5. This gives 5 < N < 17, for a total of eleven possible values.

Note that the side of length 6 is then necessarily less than the sum of the other two sides.

F89B3. In this solution, we use the identity $(\log_a b)(\log_b c)=\log_a c$, which is true for any positive values of a, b, and c other than 1.

We have $\log_{49}x^2 = 2 \log_{49}x = 2 (\log_{49} 7)(\log_7 x) =$

$= 2 (1/2) (.4) = .4$7

F89B4. We need not solve the equation directly if we note that 1/r + 1/s = (r+s)/rs = -1/12345.

F89B5. If x = Arcsin 3/5 and y = Arccos 7/25, then we must compute sin (x+y) = sin x cos y + cos x sin y. By definition, we have sin x = 3/5 and cos y = 7/25. To compute cos x, we can use the identity $\cos^2 x + \sin^2 x = 1$. We find:

$\cos^2 x + 9/25 = 1$, $\cos^2 x = 16/25$, cos x = 4/5 (x is an acute angle, by the definition of principal value).

(Another way to reach this result is to sketch a "reference triangle".)

Similarly, sin y = 24/25, so sin (x+y) =
(3/5)(7/25) + (4/5)(24/25) = (21+96)/125 = 117/125.

F89B6. We have $(a+bi)^2 = a^2 - b^2 + 2abi = -9 + 40i$. Equating real and imaginary parts, we have:

$a^2 - b^2 = -9$ and 2ab = 40, or ab = 20. Trial and error shows that a = 4, b = 5 works (the values a = -4, b = -5 are excluded by the conditions of the problem), so that a + b = 9.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO FALL, 1989

SOLUTIONS

F89B7. To form a diagonal of the polygon, we need to connect two vertices. There are $20 \cdot 19/2 = 190$ ways to choose two vertices to connect. But this total includes the sides of the polygon as well, so there are actually $190 - 20 = 170$ diagonals.

F89B8. We have $1 - 2\sin^2 x = \sin x$. Letting $s = \sin x$, this means $2s^2 + s - 1 = (2s - 1)(s+1) = 0$, and $s = 1/2, -1$. The smallest positive value of $x$ is $\pi/6$.

OR: If $\sin A = \cos B$, then $A + B = \pi/2$. Here, $x+2x = 3x = \pi/2$, and $x = \pi/6$.

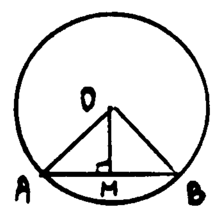
F89B9. Radians:Degrees = $\pi$: 180, so a radian measure of 1 corresponds to a degree-measure of $180/\pi$.

F89B10. We need to write the given expression in terms of the sum and product of the roots of the equation. To this end, we have:

$r^3s + rs^3 = rs(r^2 + s^2)$,
$rs = 1/3$, and
$r^2 + s^2 = (r+s)^2 - 2rs$.

Thus $r^3s + rs^3 = (1/3)(49/9 - 2/3) = (1/3)(43/9) = 43/27$.

F89B11. If the chord is AB and the center O (see diagram), then the radius OA is the hypotenuse of right triangle OAM, so $OA^2 = OM^2 + AM^2 = 25 + 25 = 50$, and $OA = \sqrt{50} = 5\sqrt{2}$.




F89B12. We can solve this problem by measuring <u>distance</u> around the clock's face in minutes. If the minute hands moves x minutes past its position at 12, the hour hand will have moved x/12 minutes. Since 60 degrees is 10 minutes, we have either (i) $x - x/12 = 10$, and $x = 120/11$ minutes
or (ii) $x - x/12 = 50$, and $x = 600/11$ minutes.

A quick sketch of the clock face will show why two positions are possible. In grading answers, students must have both values to get credit. Allow credit for exact numerical equivalents (but not rounded or truncated decimal approximations).

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER THREE FALL, 1989

SOLUTIONS

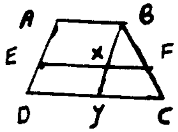
F89B13. If the polygon has n sides, then there are $n(n-1)/2 - n$ diagonals (see problem F89B7). This expression is equal to 119, so we have: $n(n-1) - 2n = 238$, or

$$n^2 - 3n = n(n-3) = 238.$$

Since $238 = 2 \times 7 \times 17$, it is not hard to see that the solution is $n = 17$ (the solution $n = -14$ is extraneous).

F89B14. This problem can be done directly. Or, we can use the fact that in an arithmetic progression the sums of terms which are placed "symmetrically" about the middle term are equal. If the progression has an even number of terms, the "middle" term must be thought of as between the second or third. Hence, the required sum is $2117 + 5 = 2122$.

F89B15. In the figure, BY is drawn parallel to AD, intersecting EF at X. Then $EX = AB = 7$, and from similar triangles BXF, BYC we find $XF:YC = BF:BC = 3:5$. Since $CY = 10 - 7 = 3$, we have $XF = 9/5$. Hence $EF = 7 + 9/5 = 44/5$.




The same answer could have been obtained by taking a weighted average of AB and CD:

$$EF = (2AB + 3CD)/5 = (14 + 30)/5 = 44/5.$$

Can you state a general theorem about this situation?

F89B16. The following solution rests on the fact that if M has remainder a, and N has remainder b when divided by 7, then MN has the same remainder as ab when divided by 7.

Since $10^2 = 100$ has remainder 2 when divided by 7, it follows that $10^6 = (10^2)^3$ has the same remainder as $2^3 = 8$ when divided by 7. This remainder is 1. Therefore, 10 raised to any power divisible by 6 will have a remainder of 1. When 1989 is divided by 6, the quotient is 331 and the remainder is 3. Hence $10^{1989}$ has the same remainder as $10^3$ when divided by 7. By direct division (or otherwise), this remainder is 6.

This "arithmetic of remainders" is important in number theory, and is called modular arithmetic.

F89B17. The multiples of 7 form an arithmetic progression, so we can add them easily, once we locate the first and last element. Since $14 < 100/7 < 15$ and $142 < 1000/7 < 143$, the first term of the progression is $15 \times 7 = 105$, while the last term is $142 \times 7 = 994$. The number of terms is given by $142 - 15 + 1 = 128$. Finally, the sum of an arithmetic progression is equal to the product of the number of terms and the average of the terms (which is also the average of the first and last term). Here, we have $(128/2)(105+994) = 70336$.

F89B18. A line and a parabola intersect, in general, at two points, since simultaneous solution of the two equations will give two roots. If the line is tangent to the parabola, it means that the two roots coincide: the equation has a double root. In the present case, we have:

$x^2 = mx + b$, or $x^2 - mx - b = 0$, which has a double root when its discriminant is zero, or when $m^2 = -4b$. Now the line $y = mx + b$ passes through the point $(1,-5)$, so that $-5 = m + b$, or $b = -m-5$. Hence we have $m^2 = 4m + 20$, or $m^2 - 4m - 20 = 0$. The larger of the two roots of this equation is $2 + 2\sqrt{6}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR FALL, 1989

SOLUTIONS

F89B19. Nine pieces of candy is 3/4 of a dozen, so the dozen would cost (4/3)(33) = 44 cents under these conditions. Since this is 5 cents more than the actual cost, a dozen pieces must cost 39 cents.

F89B20. If the other leg is b, and the hypotenuse is c, then $40^2 = 1600 = c^2 - b^2 = (c+b)(c-b)$. We can get the solutions by considering factors of 1600, letting the larger of two factors equal c+b and the smaller equal c-b (since b must be positive). It follows that c is half the sum of the factors under consideration, so that the factors must have the same parity (both odd or both even). Since 1600 is an even number, the factors must both be even. Some algebra will show that this necessary condition is also sufficient.

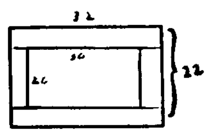
Now $1600 = 2^6 5^2$, so the number has 7x3 = 21 divisors (note that the number of divisors is odd, since 1600 is a perfect square). Of these, only 1, 5, and 25 are odd, so we must omit these and the factors they go with in counting pairs of even divisors. We must also omit the pair of factors (40,40), which does not lead to a solution. Altogether, then, there are seven pairs of even factors. These give rise to seven non-congruent triangles, with legs of length 40.

F89B21. There are nine choices for Susanna's integer, and if she chooses an odd integer, the value of the polynomial will be even, and therefore not a prime (it certainly will not be 2--the only even prime!).

Trying the remaining values, we find $6^2 - 29 = 7$; $8^2 - 29 = 35$; $10^2 - 29 = 71$; $12^2 - 29 = 115$; $14^2 - 29 = 167$. Of these, only three (7, 71, 167) are prime. The required probability is therefore 3/9 = 1/3.

F89B22. We have: $a + b + 2\sqrt{ab} = 5 + 2\sqrt{6}$, so that a + b = 5 and ab = 6. By inspection, the solutions are (a,b) = (3,2) or (2,3). Since the equations would lead to a quadratic in a or b if multiplied out, these must be the only solutions. The problem requires only (2,3).

F89B23. The border can be dissected into four long rectangles (see diagram). Two of the rectangles have dimension 1x32 square meters, and two have dimension 1x20 square meters. The total area is 104 square meters.




F89B24. If the numbers are a/r, a, and ar, then we have:

a/r + a + ar = a(1/r + 1 + r) = 21.

r/a + 1/a + 1/ra = (1/a)(r + 1 + 1/r) = 7/12.

Dividing the first equation by the second, we have a/(1/a) = $a^2$ = 21/(7/12) = 36, so a = 6 or -6. We can now solve for r:

If a = 6:

r + 1 + 1/r = 21/6 = 7/2
r + 1/r = 5/2
$2r^2 - 5r + 2 = 0$
(2r - 1)(r - 2) = 0
r = 1/2, 2

If a = -6:

r + 1 + 1/r = -21/6 = -7/2
r + 1/r = -9/2
$2r^2 + 9r + 2 = 0$
$r = (1/4)(-9 \pm \sqrt{65})$
(using the quadratic formula)

If a = 6, the solutions are 3, 6, 12 or 12, 6, 3.

If a = -6, the solutions are more difficult to express, but are not extraneous.

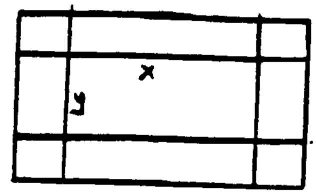
Note: In grading the contest, all four answers must be present, but in any order at all.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE FALL, 1989

## SOLUTIONS

F89B25. The area of the triangle is $10\times9/2 = 45$ $cm^2$. If the required altitude is h, then $12h/2 = 6h = 45$, and $h = 45/6 = 15/2$ centimeters.

F89B26. Suppose the dimensions of the picture alone are x and y. The diagram shows the frame dissected into eight rectangles, of areas x, y, x, y, 1, 1, 1, and 1. We then have $2x+2y + 4 = 100$, or $2x + 2y = 96$. This is the perimeter of the picture alone (its value is determined uniquely by the conditions of the problem).




F89B27. Adding the three given equations, we find that $2x + 2y + 2z = 10$, so that $x + y + z = 5$. Subtracting each of the given equations in turn from this derived relation, we find that $z = 0$, $x = 12$, and $y = -7$.

F89B28. If the number p, q, r, s, t form a geometric progression, then (for some value of a and r), they can be expressed respectively as a, ar, $ar^2$, $ar^3$, and $ar^4$. If we take logarithms to the base 77 (or any other base), we find that they form an arithmetic progression: log a, log a + log r, log a + 2 log r, log a + 3 log r, log a + 4 log r.

Since the middle term of an arithmetic progression is the arithmetic mean of the first and last terms, the value of $\log_{77}r$ is $44/2 = 22$.

F89B29. If the first marble chosen is black, then the probability that the second marble being black is clearly 1. If the first marble chosen is white, then the probability that the second marble will be black is 12/13. Since these events are equally likely (they each occur with probability 1/2), the probability that the second marble will be black is $(1/2)(12/13)+ 1/2 = 25/26$.

F89B30. We have $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$, so $a^3 = b + 3axy$, and $xy = (a^3-b)/3a$.
Now $x^2+y^2 = (x+y)^2-2xy$, so
$x^2 + y^2 = a^2 - 2(a^3-b)/3a$, or $x^2+y^2 = (a^3+2b)/3a$.
Other mathematically equivalent answers should be given credit, so long as they are given in the form of an equation.