NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE FALL, 1989

Part I: 10 Minutes

F89S1. A dealer buys cars from the factory at a price which is 20% less than the sticker price. He sells the same cars at a price equal to the sticker price. What percent profit does he make on these cars?

F89S2. The real numbers x and y satisfy

$$(2x-5)^2 + (4y+3)^2 = 0.$$

Compute $x + y$.

Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89S3. It is not hard to see that a convex polygon with more than three sides cannot have interior angles which are all acute. But what is the largest number of acute interior angles that such a polygon can have?

F89S4. Two trains approach each other along parallel tracks. One is moving at a speed of 40 miles per hour, and the other at a speed of 60 miles per hour. A passenger seated in the slower train notes that it takes 8 seconds for the faster train to completely pass his seat. How long, in miles, is the faster train?

Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89S5. At how many times between noon and midnight (twelve hours later) are the hands of a clock mutually perpendicular?

F89S6. Carol chose an ordered pair (x,y) of integers, and she did not choose zero for y. The sum, product, difference and quotient of her two integers, all taken in the same order, add up to 735. Find all such ordered pairs of integers.

ANSWERS

1. 25 or 25%
2. 7/4
3. 3
4. 2/9 or 2/9 miles
5. 22
6. (90,6) and (-120, -8) two ordered pairs required.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER TWO FALL, 1989

Part I: 10 Minutes

F89S7. A committee of 24 senators votes on a bill. Each votes "yes" or "no". The bill will pass when the number of "yes" votes is at least twice the number of "no" votes. Ten of the senators intend to vote "no". At least how many of these must change their minds for the bill to pass?

F89S8. The integer 398161 is a perfect square. What is the next largest perfect square?

Part II: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89S9. For all real numbers x, $f(x+3) = x^2 + 5x$. Write an equation expressing $f(x)$ explicitly in terms of x.

F89S10. In right triangle ABC, CH is the altitude to hypotenuse AB. The radius of the circle inscribed in triangle BCH is twice the radius of the circle inscribed in triangle ACH. If AB = 5, compute the length of AC.

Part III: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89S11. One root of the equation $x^2 - (2+2i)x + 4i = 0$ (where i represents the imaginary unit) is 2i. What is the other root of this equation?

F89S12. Line m intersects two opposite sides of a square with side 3, forming a 60° angle with each of these sides. Consider the set of line segments whose endpoints lie on the perimeter of the square, and which are all parallel to line m. What is the length of the path consisting of all the midpoints of these line segments?

ANSWERS

7. 2

8. 399424

9. $f(x) = x^2-x-6$
equation required.

10. $\sqrt{5}$

11. 2

12. $3 + \sqrt{3}$
or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE FALL, 1989

Part I: 10 Minutes

F89S13. Compute the average (arithmetic mean) of 5 and the average (arithmetic mean) of 10 and 12.

F89S14. The digits 3, 4, 5, and 6 are used, once each, to form 24 four-digit decimal numerals. These are listed in increasing order. What is the 13^{th} numeral in this list?

Part II: 10 Minutes NYCIML CONTEST THREE FALL, 1989

F89S15. A right circular cone and right circular cylinder have equal heights and equal volumes. Compute the ratio of the radius of the base of the cylinder to the radius of the base of the cone.

F89S16. The quadratic equation $ax^2 + bx + c = 0$, where x is the variable, has roots which are the squares of the roots of the equation $x^2 + x + 1 = 0$. Find the ordered triple (a,b,c) of integers.

Part III: 10 Minutes NYCIML CONTEST THREE FALL, 1989

F89S17. Compute the square of the number

$$2^{\sin \pi/6} - 2^{-\sin \pi/6}$$

F89S18. In triangle ABC, BC = 9, AB = 12 and angle ABC measures 60 degrees. Rectangle WXYZ is inscribed in the triangle, so that side WX is interior to segment BC and diagonal WY is as short as possible. Compute the length of WY.

ANSWERS

13. 8

14. 5346

15. $\sqrt{3}/3$ or equivalent

16. (1,1,1) ordered triple required

17. 1/2

18. $18/\sqrt{7}$ or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION   CONTEST NUMBER FOUR   FALL, 1989

Part I: 10 Minutes

F89S19. How many different sets of positive integers (consisting of at least <u>two</u> elements) add up to 7, if no element of the set can be used as an addend more than once?

F89S20. If x is a positive real number, and

$$a = \log_3(\log_{27}x)$$

$$b = \log_{27}(\log_3 x),$$

write an equation expressing a explicitly in terms of b.

Part II: 10 Minutes   NYCIML CONTEST FOUR   FALL, 1989

F89S21. The square of the complex number a + bi (where a and b are <u>positive integers</u>) is 2i. Find the ordered pair (a,b).

F89S22. Compute the number of cubic units in the volume of a regular tetrahedron with edge 1.

Part III: 10 Minutes   NYCIML CONTEST FOUR   FALL, 1989

F89S23. Compute the numerical value of the product

$$(\sin 15^\circ)(\cos 15^\circ).$$

F89S24. The natural number N is represented by a two-digit decimal numeral. Its square is represented by a four-digit decimal numeral. The tens digit of N^2 is identical to the tens digit of N, and the units digits of the two numbers are also identical. Find N.

ANSWERS

19. 4	21. (1,1)	23. 1/4
20. a = 3b −1 equation required	22. $\sqrt{2}/12$	24. 76

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION   CONTEST NUMBER FIVE   FALL, 1989

Part I: 10 Minutes

F89S25. Susanna took a sequence of examinations in Greek and in Latin. Her average (arithmetic mean) on all the exams was 88. Her average on the Greek exams alone was 80, and her average on the Latin exams alone was 92. What was the ratio of the number of Greek exams she took to the number of Latin exams she took?

F89S26. John's watch gains 10 minutes each hour. Bob's watch loses ten minutes each hour. They synchronize watches at midnight, and check them against each other every hour. (If they disagree, they have a ten minute argument about who is right, while the watches keep on ticking.) How many hours will elapse before their two watches again agree?

Part II: 10 Minutes   NYCIML CONTEST FIVE   FALL, 1989

F89S27. Recall that we found, in problem F89S22, that the volume of a regular tetrahedron with unit edge is $\sqrt{2}/12$. What is the length, in linear units, of the edge of a regular tetrahedron whose volume, in cubic units, is equal to its total surface area, in square units?

F89S28. In a certain sequence of natural numbers, each term after the first two is the sum of the previous two terms. If the <u>tenth</u> term is 301, find the sum of the first two terms.

Part III: 10 Minutes   NYCIML CONTEST FIVE   FALL, 1989

F89S29. A regular tetrahedron has an edge of length 8. A certain plane cutting the tetrahedron forms a cross section in the shape of a square. Compute the area of this square.

F89S30. Compute the numerical value of

$$\cos 2\pi/5 + \cos 4\pi/5 .$$

ANSWERS

25. 1:2 or equivalent	27. $6\sqrt{6}$ or equivalent	29. 16 or 16 square units.
26. 36	28. 10	30. $-1/2$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE FALL, 1989

SOLUTIONS

F89S1. If the sticker price is S, then the dealer paid .8S for each car, so his profit is S - .8S = .2S. This is 100 X .2S/.8S percent, or 25%, of his own cost.

But the problem is solved most simply by considering a single car with a sticker price of $100.

F89S2. Since squares cannot be negative, the two squares can sum to zero only if each is itself equal to zero. This makes $x = 5/2$, $y = -3/4$, and $x + y = 7/4$.

F89S3. The exterior angles of any convex polygon add up to

360°. This sum is exceeded if four exterior angles are required to be obtuse. Hence, at most three exterior angles can be obtuse. Since an interior angle is acute if and only if its adjacent exterior angle is obtuse, there can be at most three acute interior angles in any convex polygon. An acute triangle furnishes an example of how this maximum can actually be attained.

F89S4. We may think of the slower train as stationary, while the faster train approaches it at a speed of 40 + 60 = 100 miles per hour. Then the faster train travels its own length in 8 seconds, so its length is equal to eight times its speed in miles per second. Since there are 3600 seconds in an hour, this length is equal to $100 \cdot 8/3600 = 2/9$ miles (or about 1173 feet: a very long train).

F89S5. The hands are perpendicular twice during each hour, except if they are perpendicular on the hour, in which case the hand are perpendicular only once more during the hour, and only once during the previous hour. These exceptions occur at 3:00 and 9:00, so the hands are perpendicular altogether a total of 24 - 2 = 22 times.

F89S6. If the ordered pair is (a,b), then we have $(a+b) + (a-b) + ab + a/b = 2a + ab + a/b = 735$. The addend a/b is troublesome, so we factor it out:

$(a/b)(2b + b^2 + 1) = (a/b)(b+1)^2 = 735$, and $a(b+1)^2 = 735b$. Since b and b+1 are relatively prime, it follows that $(b+1)^2$

must divide 735. Since $735 = 3 \cdot 5 \cdot 7^2$, b+1 = 7 or -7, and b = 6 or -8. These possibilities lead to the two ordered pairs (90,6) and (-120,-8).

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER TWO FALL, 1989

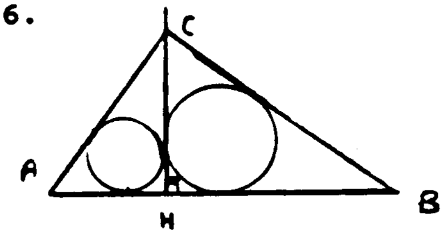
SOLUTIONS

F89S7. The condition that the bill pass is equivalent to the condition that at least 2/3, or 16, of the senators senators vote "yes". As it stands, only 14 will vote "yes". Two senators must change their minds for the bill to pass.

F89S8. The given number is between 600^2 and 700^2. A brief search will show that it is 631^2. The next largest perfect square is 632^2.

An easy way to calculate the answer, rather than multiplying out, is to note that the difference of consecutive squares is the sum of their square roots. Hence the answer to the problem is $398161 + 631 + 632 = 399424$.

F89S9. Let $y = x+3$, so that $x = y-3$. Then $f(y) = (y-3)^2 + 5(y-3) = y^2 - y - 6$. Hence $f(x) = x^2 - x - 6$.

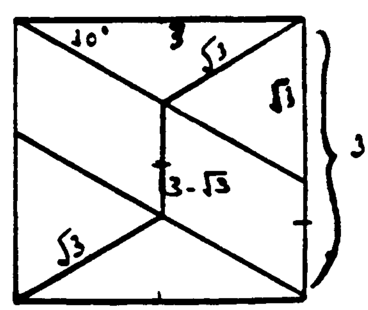


F89S10. Triangles BCH and ACH are similar, and their ratio of similitude is the ratio of their inradii, or 2:1. Hence $HC = 2AH$, and $BH = 2CH$, so that $BH = 4AH$. It follows that $AH = 1$, $CH = 2$, $BH = 4$, and $AC = \sqrt{5}$.

F89S11. The sum of the roots of the given equation is $2+2i$. Since one root is $2i$, the other root must be 2. As a check, the product of the two roots is $4i$, which can also be read from the given equation.

Note that the usual theorem that "complex roots come in conjugate pairs" applies only if the equation has <u>real</u> coefficients.

F89S12. In the diagram on the right, line m is not shown. The important segments, drawn in, are those passing through two vertices of the square and parallel to line m. These two lines cut off 30-60-90 triangles, whose sides can be found easily. The part of the path which is parallel to two sides of the square can be computed by subtracting a side of the square from the shorter side of the triangle. The other two parts of the path, which are equal to each other, are medians of the same triangle. Since the median of a right triangle is half the hypotenuse, their lengths are also not hard to find. Calculations are shown briefly in the diagram.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION   CONTEST NUMBER THREE   FALL, 1989

SOLUTIONS

F89S13. The average of 10 and 12 is 11, and the average of 5 and 11 is 8.

F89S14. There are six numerals starting with 3, six starting with 4, six starting with 5, and six starting with 6. Clearly, they appear in exactly this order on the list. Hence, the first twelve numerals start with the digits 3 and 4, and the thirteenth is the smallest starting with the digit 5. It is not hard to see that this must be 5346.

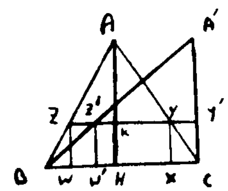
F89S15. If the radius of the cylinder is r, that of the cone is s, and the common height is h, then $\pi r^2 h = (1/3)\pi s^2 h$, or $r^2 = s^2/3$, so that $r^2 : s^2 = 1:3$, and $r:s = 1:\sqrt{3} = \sqrt{3}:3$.

F89S16. Let $y = x^2$. We must form an open sentence in y, based on the one we are given in x. Writing $x = \sqrt{y}$, we find that $y + \sqrt{y} + 1 = 0$, or $\sqrt{y} = -1-y$, or $y = y^2 + 2y + 1$, which leads to $y^2 + y + 1 = 0$. Selecting $x = -\sqrt{y}$ leads to the same result.

Note that this equation is identical with the one we started with. The roots of the equation (they're both complex) are the squares of <u>each other</u>. These roots also satisfy the equation $x^3 - 1 = 0$, so they are the "cube roots of unity". For much more about these interesting numbers, see Hall and Knight.

F89S17. If $x = 2^{\sin \pi/6}$, then the given number has the form $x - 1/x$, so its square is $x^2 + 1/x^2 - 2$. Since $\sin \pi/6 = 1/2$, $x^2 = (2^{1/2})^2 = 2$, so $1/x^2 = 1/2$, and the required number is $2 + 1/2 - 2 = 1/2$.

F89S18. If ABC were a right triangle, with leg BC, the diagonal of inscribed rectangle WXYQ is just a line from vertex C to hypotenuse AB. The minimal diagonal is thus the altitude to the hypotenuse (see diagram at right). We will reduce the general case to this observation.



For the given triangle ABC, we construct right triangle A'BC as shown at right, and think of inscribed rectangle WXYZ as "corresponding" to rectangle W'CY'Z', which is inscribed in the right triangle A'BC. The two rectangles have the same height. We will show that they have the same diagonals by showing that their lengths are equal; that is, that ZY = Z'Y'.

This last assertion follows from several pairs of similar triangles. From similar triangles AZY, ABC, ZY:BC = AK:AH (corresponding sides are in the same ratio as corresponding altitudes). From similar triangles A'Z'Y', A'BC, Z'Y':BC = A'Y':A'C = AK:AH = ZY:BC. It follows that ZY = Z'Y'.

Thus each rectangle inscribed in the original triangle corresponds to a congruent rectangle inscribed in a right triangle. Since the minimal diagonal in the right triangle is an altitude, the same length must be the minimum in the general case.

To compute this length, we note that $A'C = AH = 6\sqrt{3}$ (from 30-60-90 triangle ABH). Hence in right triangle A'BC, hypotenuse $A'B = 3\sqrt{21}$, and the altitude to $A'B = (9)(6\sqrt{3})/(3\sqrt{21}) = 18/\sqrt{7}$ (or equivalent).

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR FALL, 1989

SOLUTIONS

F89S19. By experimenting, we find that 7 is the sum of the elements of the sets {6,1}, {5,2}, {4,3}, {4,2,1}.

Note that representations such as 2+2+3 do not lead to a set of positive integers whose sum is 7, and that the singleton set {7} is excluded by the wording of the problem.

F89S20. We have $3^a = \log_{27}x = 1/(\log_x 27) = 1/(3\log_x 3)$

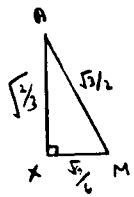
$$3^{3b} = 27^b = \log_3 x = 1/(\log_x 3),$$

so $3^a = (1/3)3^{3b} = 3^{3b-1}$, and $a = 3b - 1$.

F89S21. If $(a+bi)^2 = a^2 + 2abi - b^2 = 2i$, then (equating real and imaginary parts), $a^2 - b^2 = 0$, $2ab = 2$, and $ab = 1$. From the second equation, $a = 1$ and $b = 1$. This works in the first equation as well. Indeed, $(1+i)^2 = 2 + 2i$.

The number $-1-i$, whose square is also $2i$, does not satisfy the conditions of the problem. De Moivre's theorem gives a good picture of the situation, and an alternative solution.

F89S22. The tetrahedron can be considered as a pyramid whose base is an equilateral triangle. Its volume is then $(1/3)A_b h$, where h is the height and A_b is the area of the base. Using 30-60-90 triangles, or otherwise, it is not hard to find that the altitude of the base is $\sqrt{3}/2$, so its area is $\sqrt{3}/4$. To find the height of the tetrahedron, we pass a plane through A, X, and M, where M is the midpoint of edge BC and X is the centroid (intersection of the medians) of face BCD (see diagram).



AM is an altitude of a face of the tetrahedron, so we have already found that its length is $\sqrt{3}/2$. Now MX is part of altitude MD in equilateral triangle BCD, which is also a median of the same triangle, and point X is the intersection of the altitudes as well as the medians. Hence $MX = (1/3)MD = (1/3)AM = \sqrt{3}/6$. We can now compute AX using right triangle AMX: $AX^2 = AM^2 - MX^2 = 3/4 - 3/36 = 3/4 - 1/12 = 2/3$. The volume of the tetrahedron is thus $(1/3)(\sqrt{3}/4)(\sqrt{2}/\sqrt{3}) = \sqrt{2}/12$.

F89S23. If $P = \sin 15^\circ \cos 15^\circ$, then $2P = 2 \sin 15^\circ \cos 15^\circ = \sin 2 \cdot 15^\circ = \sin 30^\circ = 1/2$. Hence $P = 1/4$.

F89S24. We must solve, in positive integers, the equation

$N^2 - N = 100K$, or $N(N-1) = 2^2 \cdot 5^2 \cdot K$. Either N or N-1 must be even. If N is even, then it must be a multiple of 4 (since the right side of the equation is a multiple of 4, and N-1 is not a multiple of 2). Letting N = 4R (for some integer R), we have R(4R-1) = 25K, so either R or 4R-1 is a multiple of 5. If 5 divides R, then 5 does not divide 4R-1, so 25 divides R. This makes R at least 25, which makes N too large. If 5 divides 4R-1, on the other hand, 5 cannot divide R, so 25 also divides 4R-1. Examining possibilites, we find that 4R-1 must equal 75 and N = 76.

The analysis is similar if N-1 is even. We can find an integer R such that N-1 = 4R, and we find that R(4R+1) = 25K, and 25 must divide either R or 4R+1. Neither of these possibilities leads to a solution for the problem (the choice 4R+1 = 25 leads to N = 25, which is too small for the conditions of the problem).

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE FALL, 1989

SOLUTIONS

F89S25. If Susanna took G exams in Greek and L exams in Latin, then $(80G + 92L)/(G+L) = 88$, or $80G + 92L = 88G + 88L$. This leads to $4L = 8G$, so $G:L = 1:2$.

F89S26. The difference between the times shown on the watches increases by 20 minutes each hour. For the watches to show the same time, this difference must be a multiple of 12 hours. This will happen first after 36 hours.

F89S27. For a unit regular tetrahedron, the area of each face is $\sqrt{3}/4$, so the total surface area is $\sqrt{3}$. We have already found that the volume of a unit tetrahedron is $\sqrt{2}/12$. Now any regular tetrahedron is similar to a unit regular tetrahedron. If x is the edge of any regular tetrahedron, it follows that its total surface area is $x^2\sqrt{3}$, and its volume is $x^3\sqrt{2}/12$. For this problem, these two are equal, which leads to $x = 12\sqrt{3}/\sqrt{2} = 6\sqrt{6}$.

F89S28. If the n^{th} term is a_n, then $a_{10} = 301$. By the rule of formation of the sequence, this means that $a_9 + a_8 = 301$. Replacing a_9 by $a_8 + a_7$, we find that $2a_8 + a_7 = 301$.

Continuing this process, we have:

$$3a_7 + 2a_6 = 301$$
$$5a_6 + 3a_5 = 301$$
$$8a_5 + 5a_4 = 301$$
$$13a_4 + 8a_3 = 301$$
$$21a_3 + 13a_2 = 301$$
$$34a_2 + 21a_1 = 301.$$

We must solve this last equations for positive integers a_1 and a_2. Since $301 = 7\cdot 43$, all but one of the terms in the equation is a multiple of 7. Hence a_2 must also be a multiple of 7. Trying $a_2 = 7$, we find $a_1 = 3$ works, so that $a_1 + a_2 = 10$.

Larger multiples of 7 do not yield solutions in positive integers.

F89S29. It is not hard, from symmetry, to see that the plane described in the problem must be parallel to two opposite edges of the tetrahedron, and pass through the midpoints of the other four sides. Since the faces of the tetrahedron are equilateral triangles, it follows that a side of the square is half of one edge, or 4. Hence the area of the square is 16 square units.

Proofs of the assertions made above about the cutting plane are left to the contestant.

F89S30. Let $S = \cos 2\pi/5 + \cos 4\pi/5$. By the usual sum-to-product formulas,

$$S = 2(\cos 3\pi/5)(\cos -\pi/5)$$
$$= 2(\cos 3\pi/5)(\cos \pi/5),$$

since the cosine is an <u>even</u> function. We seek to relate this to the formula for the sine of twice an angle. But the sines of the angles given are lacking. We multiply by the necessary factors, hoping that something will drop out (the statement of the problem implies that something will):

$$S(2\sin 3\pi/5)(\sin \pi/5) =$$
$$= 2(\sin 3\pi/5)(\cos \pi/5)(2)(\cos 3\pi/5)(\sin \pi/5)$$
$$= (\sin 6\pi/5)(\sin 2\pi/5) = -(\sin \pi/5)(\sin 3\pi/5).$$

This chain of equalities implies that $S = -1/2$.

January 12, 1990

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1989 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	F89S6	(90, 6), (-1470,-2), (-120,-8)
	F89S16	(1,1,1) or any (k,k,k) $k \neq 0$
	F89S26	36 or 72
Junior	F89J10	= 0 should be added to the expression in the problem to make it an equation.

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML

Last one