NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE FALL, 1989

Part I: 10 Minutes

F89J1. George had a number of jellybeans. He ate 1/3 of them, and left the rest on a plate. Martha saw the jellybeans George had left, and ate some more of them. Then George came back and found that only 1/5 of the original number of jellybeans was left. What fraction of the jellybeans that Martha found did she eat?

F89J2. What is the sum of all the digits of all the positive integers (in decimal notation) less than 100?

Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89J3. Find all real values of x such that $x^2(x-1) - 4x + 4 = 0$.

F89J4. Triangle ABC has all its vertices on the perimeter of a square with unit area. What is the largest possible area that triangle ABC could have?

Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1989

F89J5. Find the largest prime number which can be written as the sum of two (positive) prime numbers and also as the difference of two (positive) prime numbers.

F89J6. Circles O and P (with centers at points O and P respectively) are externally tangent at A, and OP = 3AP. Lne CD is tangent to circle O at C and to circle P at D. Compute the degree-measure of angle CAD.

ANSWERS

1. 7/10

2. 900

3. 2, -2, 1: All three required, in any order.

4. 1/2

5. 5

6. 90 or 90°

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER TWO FALL, 1989

Part I: 10 Minutes

F89J7. Compute the numerical value of $1990^2 - 1989^2$.

F89J8. In triangle ABC, angle C measures 90 degrees. Medians AM and BN have lengths of 3 and 4 respectively. Find the length of AB.

Part II: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89J9. Find the smallest natural number larger than 1000 which is a divisor of 1112111.

F89J10. Find the largest integer value of p for which the equation $x^2 - 11x + p$ has two real roots.

Part III: 10 Minutes NYCIML CONTEST TWO FALL, 1989

F89J11. The Martian day is divided into 36 <u>koudads</u> (a measure of time which plays the same role as our <u>hour</u>). Martian clocks are similar to ours, but with 18 numerals equally spaced around the circumference. On such a clock, the Martian numeral for x is diametrically opposite the Martian numeral which represents the number 3. What is the decimal numeral for x?

F89J12. A tetrahedron is inscribed in a hemisphere of radius 12, so that three of its vertices are on the base of the hemisphere and a fourth vertex is on its curved surface. Find the largest possible volume of such a tetrahedron.

ANSWERS

7. 3979

8. $2\sqrt{5}$

9. 1001

10. 30

11. 12

12. $432\sqrt{3}$ or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER THREE FALL, 1989

Part I: 10 Minutes

F89J13. An interior diagonal of a cube is a line segment whose endpoints are vertices of the cube, and which does not lie on a face of the cube. How many interior diagonals does a cube have?

F89J14. Nancy and Ron are both invited to a dinner party. There will be six at dinner altogether, seated at a round table. Nancy and Ron cannot sit next to each other without causing a scene. All the other guests, however, are compatible with each other, and with Nancy and Ron. In how many ways can they be arranged around the table, with Nancy and Ron not seated next to each other? (Two arrangements are considered distinct if and only if even one person has a different neighbor to the right in the two arrangements).

Part II: 10 Minutes NYCIML CONTEST THREE FALL, 1989

F89J15. On Uranus, each day is divided into a (very large) number of moudads (a unit of time analagous to our hour). The clocks on Uranus are similar to our own, but with a much larger number of numerals equally spaced around the circumference. If, on such a clock, the numeral for 23 is diametrically opposite the numeral for 71, how many moudads are there on the clock?

F89J16. Peter is playing a carnival game in which he gets up to 100 chances to hit a ducky with a ball. If he hits the ducky on his first try, he wins $1. If he hits on the second try he wins $2. If he hits on the third try he wins $4. The game continues, with the reward doubling for each shot Peter takes. If he misses the ducky on a try, there is no penalty, and the reward for the next try doubles anyway. If Peter won $100, how many times did he hit the ducky?

Part III: 10 Minutes NYCIML CONTEST THREE FALL, 1989

F89J17. Write the polynomial $x^4 - 2x^3 + 2x^2 - x - 2$ as the product of two other polynomials, whose coefficients are all integers.

F89J18. Find the unique ordered 5-tuple (a, b, c, d, e) of positive real numbers such that:

$$abc = a + b + c$$
$$bcd = b + c + d$$
$$cde = c + d + e$$
$$dea = d + e + a$$
$$eab = e + a + b$$

ANSWERS

13. 4
14. 72
15. 96
16. 3
17. $(x^2-x-1)(x^2-x+2)$
18. $(\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3})$ ordered 5-tuple required

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER ONE FALL, 1989

SOLUTIONS

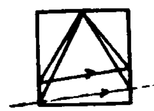
F89J1. If the fraction Martha ate was x, then $(2/3)(1-x) = 1/5$, $1-x = 3/10$, and $x = 7/10$.

F89J2. The sum of the digits from 1 through 9 is 45. The units digits of the numbers from 1 to 100 form ten copies of the digits from 1 through 9, which contributes 450 to the required sum. The tens digits of the numbers contribute ten of each digit, for another contribution of 450. The total required is 900.

F89J3. The given polynomial can be written as $x^2(x-1) - 4(x-1) = (x^2-4)(x-1)$. Therefore, its roots are given by $x^2 - 4 = 0$ and $x - 1 = 0$, so they are 2, -2, and 1.

F89J4. It is not hard to see that the largest area occurs when the vertices of the triangle coincide with three of the vertices ofthe square. In this case, the area is 1/2.

To see that this area is largest, consider first a triangle with none of its vertices on those of the square (see diagram). Clearly, the area of such a triangle can be made larger, for instance by translating one side until it intersects a vertex of the square.

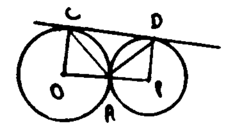


Next, consider a triangle with one vertex on a vertex of the square. As the figure shows, its area can be enlarged by placing another vertex on a vertex of the square.

A similar argument will show that the third vertex of the triangle must also be on a vertex of the square for the area to be maximal.

F89J5. The sum and difference of two odd primes is even. Hence, one of the primes in both the sum and the difference must be the prime 2. This means that we are searching for a prime which is both two less and two more than a prime. It is not hard to see that 5 is the only such prime.

F89J6. Radii OC and PD are both perpendicular to tangent CD, so that CO and DP are parallel. It follows that angles COA and DPA are supplementary. Therefore, halves of the angles are complementary. Now Angle CDA is half the measure of arc AD, so it is also half the measure of angle APD. Similarly, angle DCA is half the measure of angle COA. Therefore angles ACD, ADC are complementary, and angle CAD measures 90 degrees. The condition that OP = 3AP is irrelevant to the problem.



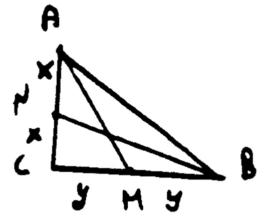
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER TWO FALL, 1989

SOLUTIONS

F89J7. We have $1990^2 - 1989^2 = (1990 + 1989)(1990-1989) = (1990+1989)(1) = 3979$.

F89J8. If $AC = 2x$ and $BC = 2y$ (see diagram), then $AB^2 = 4x^2 + 4y^2$. From right triangles ACM, BCN, we have $4x^2 + y^2 = 9$, $x^2 + 4y^2 = 16$. Adding, we find that $5x^2 + 5y^2 = 25$, so $x^2 + y^2 = 5$, $4x^2 + 4y^2 = 20$, and $AB = \sqrt{20} = 2\sqrt{5}$ (accept also $\sqrt{20}$).



F89J9. One can proceed by trial and error, or note that:

$1112111 = 1111000 + 1111 = 1111(1000 + 1) = 1111 \times 1001$.

F89J10. The discriminant of the given equation must be positive, or $121 - 4p > 0$. This leads to $p < 121/4 = 30.25$. The largest such integral value is $p = 30$.

F89J11. Clearly, the Martian numeral for 18 is opposite the numeral for 9. Therefore, the numeral for 1 is opposite the numeral for 10, the numeral for 2 is opposite the numeral for 11, and the numeral for 3 is opposite the numeral for 12.

F89J12. The volume of a tetrahedron depends on the area of its base and the length of its altitude. In this case, both are maximal if the base is an equilateral triangle inscribed in the base of the hemisphere, and if the fourth vertex is directly above the centroid of the base. In this position, the altitude of the tetrahedron is a radius of the hemisphere, and it is not hard to verify that the area of a triangle inscribed in a circle of radius 12 is $108\sqrt{3}$. Hence the required volume is $(1/3)(12)(108\sqrt{3}) = 432\sqrt{3}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER THREE FALL, 1989

SOLUTIONS

F89J13. An interior diagonal connects two vertices of the cube which are not on the same face; that is, these two vertices are opposite vertices of the cube. Each vertex has exactly one opposite vertex, so there are four pairs of opposite vertices, and four interior diagonals.

F89J14. If Nancy comes in, she may take any seat. If Ron then enters, he may take any one of three seats, since he cannot sit next to Nancy nor in the seat she has taken. The rest of the guests can take the remaining four seats in any way at all, so there are $4!=4\times3\times2\times1$ arrangements for them, and $3\times4! = 72$ arrangements in all.

Notice that we do not count the six different places that Nancy can sit, since seating her in different seats merely results in rotations of seating arrangements already counted.

F89J15. If there are M moudads on the clock, then the numeral for x is opposite the numeral for $x + M/2$. If this total is larger than M, then the numeral for x is opposite that for $x - M/2$ (try using $M = 12$ for a familiar example). Hence $71 = 23 + M/2$, and $M/2 = 48$, so $M = 96$.

F89J16. The amounts Peter wins are always powers of two, so we must represent 100 as sums of these powers (this is equivalent to representing 100 in binary notation). We have: $100 = 64 + 32 + 4$ (the representation is unique). Peter hit the ducky on his third, sixth, and seventh try.

F89J17. It is not hard to see that the given polynomial has no linear factors. Thus we must factor into two quadratic factors. Clearly, the lead coefficient of each is 1, and the constant coefficients are -1 and 2 (or 1 and -2).

Thus the two factors have the form $(x^2 + ax -1)$ and $(x^2 +cx +2)$. Multiplying out and comparing coefficients, we find that $ac = 1$, $a + c = -2$. These values, in fact, give the required factorization. The factors are unique up to order, and with the exception that both may be multiplied by -1. Credit should be given only for these alternatives.

F89J18. Subtracting the first equation from the second, we have $bc(d- a) = d - a$. If $d-a$ is not zero, we can divide by it, to find that $bc = 1$. Substituting then in the first equation, we find $a = a + b + c$, or $b + c = 0$. However, it is not difficult to see that if $b + c = 0$, bc cannot equal 1. Therefore, $d - a$ must be zero, so that $d = a$.
Reasoning similarly, we find that $b = e$, $c = a$, and $d=b$: all five unknowns must have the same value. Hence, each must be a solution of the equation $x^3=3x$. These solutions are 0, $\sqrt{3}$ and $-\sqrt{3}$. Of these, only $\sqrt{3}$ fits the conditions of the problem.

The answer must be written as an ordered 5-tuple:
$(\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3})$.
Other expressions are probably not mathematically equivalent.

January 12, 1990

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1989 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	F89S6	(90, 6), (-1470,-2), (-120,-8)
	F89S16	(1,1,1) or any (k,k,k) $k \neq 0$
	F89S26	36 or 72
Junior	F89J10	= 0 should be added to the expression in the problem to make it an equation.

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML

Last one