NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE SPRING 1989

PART I: Time 10 Minutes

S89B1. Express as a decimal numeral the number $\sqrt{9^5}$.

S89B2. If m ounces of a 10% solution of acid is mixed with m+2 ounces of a 20% solution, the result is an 18% solution of acid. Find m.

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PART II: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B3. The integer x is greater than 1, and is the smallest one for which there exists an integer y, and such that $\sqrt{x} = \sqrt[3]{y}$. Compute the numerical value of x + y.

S89B4. Points A, B, C, and D are on consecutive sides of a square with side 6, and quadrilateral ABCD is a rectangle. Find the perimeter of rectangle ABCD.

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PART III: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B5. Find the smallest natural number N > 1 such that the sum
1 + 2 + 3 + . . . + N is a perfect square.

S89B6. If $a = \log_4 32$ and $b = \log_8 16$,

compute the ratio a:b in simplest form.

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ANSWERS

1. 243
2. 2/3 or equivalent
3. 12
4. $12\sqrt{2}$
5. 8
6. 15:8 or 15/8

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO SPRING 1989

PART I: Time 10 Minutes

S89B7. For real or complex numbers such that $z^2 \neq 1$, $f(z) = \dfrac{z^2 + 1}{z^2 - 1}$. If i represents the imaginary unit, compute f(i).

S89B8. For all real values of x, find the greatest possible value of the expression $x - x^2$.

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PART II: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B9. If q is a positive real number, and $q^{1/3} = 2$, compute $q^2$.

S89B10. How many distinct points do the graphs of the equations $x^2 + y^2 = 16$ and $y = x^2 - 4$ have in common?

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PART III: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B11. In a certain arithmetic progression, the first term is 4, the $N^{th}$ term is 36, and the sum is 600. Compute N (the number of terms being added).

S89B12. If $\sin x/2 = 3/5$, compute $\cos 2x$.

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ANSWERS

| | | |
|---|---|---|
| 7. 0 | 9. 64 | 11. 30 |
| 8. 1/4 | 10. 3 or three | 12. $-527/625$ |

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER THREE SPRING 1989

PART I: Time 10 Minutes

S89B13. In Guacamora, the probability that it will rain each day is 2/3. If it does not rain on Monday, what is the probability that it will also not rain both on Tuesday and Wednesday?

S89B14. Circles O and P are externally tangent at point X. If OX = 3, and PX = 5, find the length of the common external tangent segment for the two circles.

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PART II: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B15. If Arcsin denotes principal value, compute
sin (Arcsin 3/5 + Arcsin 4/5).

S89B16. For how many real values of x does the expression

$$\frac{2}{1 - \frac{1}{x-3}}$$ have no meaning?

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PART III: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B17. For all real values of x, let f(x) = 1-x, and let g(x) = 1/x. Find all real values of y such that f(g(f(g(y)))) = 3.

S89B18. Find all real numbers x such that
$|3x - 2| + 2x - 3 = 7$.

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ANSWERS

13. 1/9
14. $2\sqrt{15}$
15. 1
16. 2 or 'two'
17. 2/3
18. 12/5, -8
both required

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR SPRING 1989

PART I: Time 10 Minutes

S89B19. If the polynomial $2x^2 - ax + b$ has the factors $2x+1$ and $x-2$, compute the value of $a + b$.

S89B20. In rectangle ABCD, AB = 9 and BC = 16. Points E and F trisect diagonal AC. Compute the area of triangle BEF.

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PART II: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B21. If $x = 2^k + 1$ and $y = 2^{-k} + 1$, write an equation expressing y explicitly in terms of x.

S89B22. Compute the area of a rectangle whose diagonal is 10 units long, and whose width is five times as long as its length.

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PART III: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B23. Find all real numbers x such that

$x^4 - 13x^2 + 36 = 0.$

S89B24. In an isosceles triangle, the lengths of the base and the median to one leg are equal. Find the cosine of the vertex angle.

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ANSWERS

| | | |
|---|---|---|
| 19. 1 | 21. $y = (x)/(x-1)$ or equivalent equation | 23. 3, -3, 2,-2: all four required |
| 20. 24 | 22. 250/13 or equivalent | 24. 3/4 |

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE SPRING 1989

PART I: Time 10 Minutes

S89B25. If $a = \sqrt{2 + \sqrt{2}}$, and $b = \sqrt{2 + a}$, compute the numerical value of $b^2 - a$.

S89B26. A train, x meters long, traveling at a constant speed, takes 20 seconds from the time it first enters a tunnel 300 meters long until the time it completely emerges from the tunnel. One of the stationary ceiling lights in the tunnel is directly above the train for 10 seconds. Find the value of x.

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PART II: Time 10 Minutes NYCIML B DIVISION SPRING 1989

S89B27. On hypotenuse AB of right triangle ABC, D is the point for which CB = BD. If angle B measures 40 degrees, find the degree measure of angle ACD.

S89B28. Find all real values of x for which

$$\sqrt{x^2 - 2x + 1} - 2 = \frac{3}{\sqrt{x^2 - 2x + 1}}.$$

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PART III: Time 10 Minutes NYCIML B DIVISION SPRING 1989

5S89B29. Express the number $\sqrt{313^2 - 312^2}$ as a positive integer in the usual decimal notation.

S89B30. The numerals 1, 2, 3, 4, . . . . , 9999, 10000 are each written down once. What is the sum of all the digits used to write these numerals?

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ANSWERS

| | | |
|---|---|---|
| 25. 2 | 27. 20 | 29. 25 |
| 26. 300 | 28. 4, -2 both required | 30. 180001 |

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER ONE SPRING 1989

SOLUTIONS

S89B1. Since $9^5 = (3^2)^5 = 3^{10}$, $\sqrt{9^5} = \sqrt{3^{10}} = (3^{10})^{1/2} = 3^5 =$ 243.

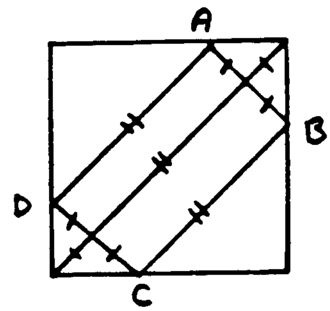
S89B2. The amounts of acid in the two original solutions are .1m and .2(m+2) respectively. Hence the concentration of acid in the end solution is

$$\frac{.1m+.2(m+2)}{2m+2} = .18$$

This leads to $.3m + .4 = .18(2m+2)$, and $m = 2/3$.

S89B3. If $\sqrt{x} = \sqrt[3]{y}$, then (raising each side to the sixth power), $x^3 = y^2$. A number which is both a perfect square and a perfect cube is a sixth power. The smallest sixth power (greater than one) is $2^6 = 64$. This observation leads to $x = 4$ and $y = 8$, so that $x+y = 12$.

S89B4. Noting the equal segments as marked in the diagram, we see that half of the perimeter of the rectangle is equal to a diagonal of the square. Hence the entire perimeter is equal to twice that diagonal, or $12\sqrt{2}$.




S89B5. By direct calculation, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =$ $= 36$, which is the square of 6.

S89B6. We have $4^a = 32$ or $2^{2a} = 2^5$ and therefore $a = 5/2$.

We have $8^b = 16$ or $2^{3b} = 2^4$ and therefore $b = 4/3$.

$a:b = (5/2):(4/3) = 15:8$.

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO SPRING 1989

SOLUTIONS

S89B7. Since $z^2 + 1 = i^2 + 1 = -1+1 = 0$, the value of $f(i)$ is 0.

S89B8. The given expression is equal to $x(1-x)$. Since $x + 1-x = 1$ (a constant), the product of $x$ and $1-x$ is maximal when the two numbers are equal. This leads to $2x = 1$, or $x = 1/2$, and the maximal value of $x-x^2$ is 1/4.

S89B9. We have $q^2 = (q^{1/3})^6 = 2^6 = 64$.

S89B10. The first graph is a circle of radius 4 about the origin. The second is a parabola ($y = x^2$), moved <u>down</u> four units. There are two points of intersection and one point of tangency, making three points altogether.

An algebraic solution of the two equations will confirm this.

S89B11. The sum of N terms of an arithmetic progression is just N times the average term. The average term is also the average of the first and last term. This average, here, is 20, so $20N = 600$, and $N = 30$.

S89B12. $\sin^2 x/2 = 9/25 = (1-\cos x)/2$, so $\cos x = 7/25$.

Then $\cos 2x = 2 \cos^2 x - 1 = 2(49/625) - 1 = 98/625 - 1 = -527/625$. A decimal equivalent is also acceptable.

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE

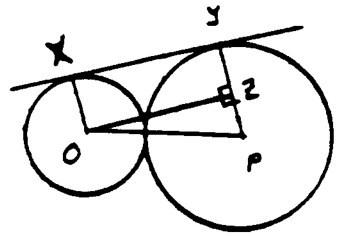
SENIOR B DIVISION CONTEST NUMBER THREE SPRING 1989

SOLUTIONS

S89B13. The probability that it will not rain on Tuesday is 1/3, and that it will not rain on Wednesday is also 1/3. Hence the probability that it will rain on neither day is 1/9.

Note that the lack of rain on Monday is not relevant to the problem, and that it does not matter if the Tuesday and Wednesday referred to are not consecutive.

S89B14. Draw $OZ \perp YP$ (see diagram). Then, in right triangle OZP, $OZ^2 = OP^2 - ZP^2 = 64 - 4 = 60$. Finally, in rectangle XYZO, $XY = OZ = \sqrt{60} = 2\sqrt{15}$.




S89B15. If P = Arcsin 3/5, then sin P = 3/5, so that cos P = 4/5. If Q = Arcsin 4/5, then sin Q = 4/5 = cos P. But this means that the two angles are complementary, so P + Q = Arcsin 3/5 + Arcsin 4/5 = $\pi/2$, and sin $\pi/2$ = 1.

S89B16. Certainly, if x = 3, the small fraction 1/(x-3) is meaningless. The expression is also meaningless if the denominator of the larger fraction is zero; that is, if 1/(x-3) = 1, or x = 4. Thus there are two such values of x.

S89B17. We have g(y) = 1/y

f(1/y) = 1 - 1/y = (y-1)/y

g( (y-1)/y ) = y/(y-1)

f( y/(y-1) ) = 1 - y/(y-1) = -1/(y-1).

This last expression must equal 3. Then y-1 = -1/3, so y = 2/3.

S89B18. We can rewrite the given equation as $|3x-2| = 10 - 2x$.

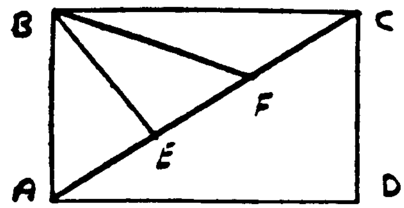
If $x \geq 2/3$, this is equivalent to 3x - 2 = 10 - 2x, which has the solution x = 12/5.

If x < 2/3, this is equivalent to 2 - 3x = 10 - 2x, which has the solution x = -8.

NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR SPRING
1989

SOLUTIONS

S89B19. Since $(2x+1)(x-2) = 2x^2 - 3x - 2$, **a = 3 and b = -2**, so $a+b = 1$.

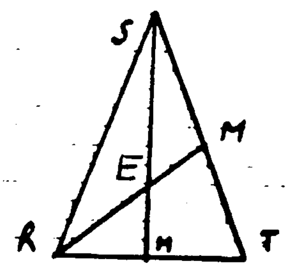



S89B20. In the diagram, triangle BEF has the same altitude as triangle ABC, and one-third the base. Its area is thus 1/3 that of triangle ABC, which makes it one-sixth that of the entire rectangle. The area of the rectangle is 144, so the area of triangle BEF is 24.

S89B21. We have $x - 1 = 2^k$, so $1/(x-1) = 1/2^k = 2^{-k}$, and $y = 1/(x-1) + 1 = x/(x+1)$. Equivalent <u>equations</u> are acceptable.

S89B22. If the length is x, then the width is 5x. From the Pythagorean theorem, $x^2 + (5x)^2 = 100$, or $26x^2 = 100$. Then $x^2 = 100/26 = 50/13$. But the area is $5x^2$, which is 250/13.

S89B23. Let $y = x^2$. Then $y^2 - 13x + 36 = (y - 9)(y-4) = 0$, so $y = 9$ or 4. It follows that $x = 3, -3, 2, -2$. All four values are required for credit.




**S89B24. If the triangle is RST, with vertex angle S (see diagram), then we can draw altitude SH. Isosceles triangles MRT, TSR are similar, so angle RST = angle TRM. Now HR = RT/2 and RE = 2RM/3 = 2RT/3 (since medians divide each other in the ratio 2:1). Thus cos <RST = cos <TRM = cos <HRE = (RT/2)/(2RT/3) = 3/4.**

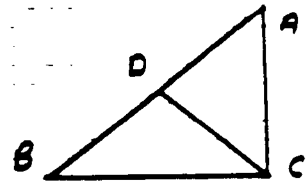
NEW YORK CITY SENIOR HIGH INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE SPRING 1989

SOLUTIONS

S89B25. Since $b^2 = 2 + a$, $b^2 - a = 2$.

S89B26. The train travels 300 + x meters in 20 seconds. Since the light is above the train for 10 seconds, the train travels x meters in 10 seconds, so it wuld travel 2x meters in 20 seconds. Thus, $300+x = 2x$, and $x = 300$.

S89B27. In isosceles triangle BCD, $\angle BCD = \angle BDC = 70$ degrees, so $\angle ACD = 90 - 70 = 20$.




S89B28. Let $y = \sqrt{x^2 - 2x + 1}$. Then $y - 2 = 3/y$, which leads to the quadratic $y^2 - 2y - 3 = 0$. This quadratic has roots 3, -1.

Thus $\sqrt{x^2 - 2x + 1} = 3$, since -1 is not a possible value for this radical. This could be solved as a second quadratic, or we could note that $\sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = |x-1|$. The roots of this equation are $x = 4, -2$.

S89B29. We have $313^2 - 312^2 = (313+312)(313-312) = 625 \times 1 = 625$, and $\sqrt{625} = 25$.

S89B30. We can visualize these numerals as written in a column, and all their places filled in with zeroes, so as to make them all four-digit numbers. The inclusion of zeroes will not change the sum of the digits:

0000 (we include this to make the sum easier. It does
0001 not affect the digit-sum. We also exclude 10000.
0002 This reduces our answer by one, which we will add
0003 back later on)
....
....
....
0010 Here, in each column, each digit will be used the
0011 same number of times. Since there are 10000
0012 numbers, each of the ten digits is used 10000/10 =
... 1000 times in each of the four columns. Thus the
0100 sum of all the digits to the left is equal to
0101 $4 \times 1000 \times (1+2+3+4+5+6+7+8+9+0) = 180000$. Adding
0102 in the digit "1" (from the numeral 10000, which we
... omitted), we find the answer to be 180001.
...
...
9999

June 12, 1989

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1989 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

| | Question | Correct answer |
|---|---|---|
| Senior A | S89S1 | 885 |
| | S89S6 | 5/6 or 2/3 |
| | S89S9 | 100 |
| Senior B | S89B4 | This question was eliminated from the competition. If the rectangle is considered a square there are infinite solutions. |

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML