NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION   CONTEST NUMBER ONE   SPRING, 1989

Part I: 10 Minutes

S89S1. The number x is the largest number whose decimal numeral contains three distinct digits. The number y is the smallest number (greater than 100) whose decimal numeral contains three distinct digits. Compute x - y.

S89S2. A mile-long bridge is to be subdivided into smaller measures, called Smoots and Knuts. A Smoot is 1/6 of a mile, while a Knut is 1/8 of a mile. Marks are made at the end of each Smoot and also at the end of each Knut, but only one mark is made if the endpoints of the two measurements coincide. Thus, for instance, a single mark is made at the beginning of the bridge, and another single mark is made at the end. Including these two marks, how many marks are made on the bridge?

Part II: 10 Minutes   NYCIML CONTEST ONE   SPRING, 1989

S89S3. If x is a real number, find the smallest possible value of the expression $1/\sqrt{1+x} + 1/\sqrt{1-x}$

S89S4. Points X and Y are interior points of rectangle ABCD (with diagonal AC). Rays AX and BX bisect angles DAB and ABC respectively, while rays DY and CY bisect angles ADC and BCD respectively. Line XY intersects AB at P and CD at Q. If PX = XY = YQ, compute the ratio AB:BC.

Part III: 10 Minutes   NYCIML CONTEST ONE   SPRING, 1989

S89S5. For all real numbers x, $f(x) = 1/\sqrt{1-x}$. If f(a) = 2, compute f(1-a).

S89S6. In Cartesian 3-space, a polyhedra has its seven vertices at the points (0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (0,1,1) and (1,1,1). What is the volume of this polyhedron?

ANSWERS

1. 864	3. 2	5. $2/\sqrt{3}$ or equivalent
2. 13	4. 2:3	6. 5/6

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER TWO SPRING, 1989

Part I: 10 Minutes

S89S7. Find the sum of all the even three-digit numbers (in decimal notation), which are multiples of 3.

S89S8. Rectangle ABCD is inscribed in a semicircle of radius 12, so that points A and B are on the semicircle and points C and D are on its diameter. Find the largest possible area of rectangle ABCD.

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1989

S89S9. <u>How many</u> positive integers do <u>not</u> satisfy the inequality

$$x^{\log \sqrt{x}} > x$$, (where the base of the logarithms is 10)?

S89S10. For all real numbers x, $f(2x) = x^2 - x + 3$. Write an equation expressing f(x) in terms of x.

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1989

S89S11. Circles O and P each have a radius of 4 units, and point P lies on circle O. The circles intersect at point A. The tangent to circle O at point A intersects circle P at points A and X. The tangent to circle P at point A intersects circle O at points A and Y. Find the area of triangle AXY.

S89S12. What is the smallest number of terms of the arithmetic progression
$\sqrt{11} + 20, \sqrt{11} + 16, \ldots, \sqrt{11} - 4\ldots$ which must be added before the sum is negative?

ANSWERS

7. 82350 9. 99 11. $12\sqrt{3}$

8. 144 10. $f(x) = x^2/4 - x/2 + 3$ or equivalent. 12. 13

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER THREE SPRING, 1989

Part I: 10 Minutes

S89S13. If 2^y is four times 8^x, write an equation expressing y explicitly in terms of x.

S89S14. Parallelogram ABCD is inscribed in a semicircle, so that A and B are on the diameter of the semicircle, and points C and D are on the semicircle itself. If the radius of the semicircle is 10, compute the largest possible area of parallelogram ABCD.

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1989

S89S15. Find the largest integer k such that the expression $x^2 + 8x + k$ can be factored into two linear polynomials, each with integral coefficients.

S89S16. Find the degree-measure of the smallest positive acute angle x such that $16 \sin^2 x \cos^2 x = 3$.

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1989

S89S17. All terms of a certain geometric progression are positive. If the first term is $\sqrt{2}$, and the third term is $3\sqrt{2}$, compute the second term.

S89S18. If p and q are the roots of the equation $x^2 - x + 9 = 0$, compute the numerical value of the expression $\frac{1}{p^2} + \frac{1}{q^2}$.

ANSWERS

13. $y = 3x + 2$ (equation required)

14. 100

15. 16

16. 30 or 30°

17. $\sqrt{6}$

18. -17/81

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER FOUR SPRING, 1989

Part I: 10 Minutes

S89S19. In a certain arithmetic progression, the fifth term is m and the sixth term is n. Express in terms of m and n the sum of the first ten terms of this arithmetic progression.

S89S20. If $\sin 2a = 1/7$, compute the numerical value of

$$\sin^4 a + \cos^4 a.$$

Part II: 10 Minutes NYCIML CONTEST FOUR SPRING, 1989

S89S21. Points X and Y are chosen on line segment AB such that AY = 2AX. If AB = 6, find the maximum possible value of the product $AX \cdot XY \cdot YB$.

S89S22. Find the volume, in cubic units, of a regular tetrahedron with a side equal to 12 units.

Part III: 10 Minutes NYCIML CONTEST FOUR SPRING, 1989

S89S23. The sequence $\langle a_n \rangle$ is defined by $a_n = 2n$. The sequence $\langle b_n \rangle$ is defined by $b_1 = a_1$, and $b_n = a_{b_{n-1}}$ for $n>1$. Compute the numerical value of b_{10}.

S89S24. If a, b, and c are the three roots of $x^3 - 3x + 7 = 0$, compute the numerical value of $(a+1)(b+1)(c+1)$.

ANSWERS

19. $5m + 5n$ or equivalent

20. 97/98

21. 8

22. $144\sqrt{2}$

23. 1024

24. -9

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR A DIVISION CONTEST NUMBER FIVE SPRING, 1989

Part I: 10 Minutes

S89S25. The positive integer N is a multiple of 17. The binary (base two) numeral representing N has a number of digits equal to 0, but only three digits equal to 1. If N is the smallest such integer, write the decimal numeral for N.

S89S26. Point X is one vertex of a parallelopiped. The three edges of the parallelopiped which meet at X have lengths 3, 6, and 8 units, and each pair of these edges forms an angle of 60 degrees. Compute the volume, in cubic units, of the parallelopiped.

Part II: 10 Minutes NYCIML CONTEST FIVE SPRING, 1989

S89S27. In triangle ABC, AC = 5, CB = 8 and AC is perpendicular to BC. The region S consists of all the midpoints of line segments with one endpoint on line segment AC and the other endpoint on line segment AB. Compute the area of region S.

S89S28. If x is a real number, find the maximum possible value of the expression

$$\frac{\sin^4 x + \cos^4 x - 1}{\sin^6 x + \cos^6 x - 1}$$

Part III: 10 Minutes NYCIML CONTEST FIVE SPRING, 1989

S89S29. There are 5! = 120 positive integers whose decimal numerals are formed by permutations of the digits of the numeral 12345. Find the sum of these 120 positive integers.

S898S30. Radii OX and OY of circle O both have unit length, and form an acute angle whose tangent is 2. Rectangle ABCD is inscribed in sector OXY, so that points A and B are both on radius OX, point D is on radius OY, and point C is on arc XY. Find the ratio AD:AB such that the <u>perimeter</u> of rectangle ABCD is maximal.

ANSWERS

25. 289	27. 10	29. 3999960
26. $72\sqrt{2}$	28. 2/3	30. 2:3

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER ONE SPRING, 1989

SOLUTIONS

S89S1 Since $x = 987$, and $y = 123$, $x - y = 864$.

S89S2. For the Knuts alone, 9 marks are necessary, and for the Smoots alone, 7. We must subtract the number of marks which coincide. That is, we must subtract the number of ordered pairs (a,b) such that $0 \leq a \leq 8$, $0 \leq b \leq 6$, and:

$$a/8 = b/6, \text{ or } 6a = 8b, \text{ or } 3a = 4b.$$

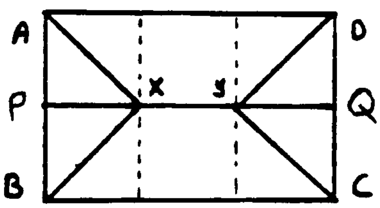
By direct computation, we find that $(a,b) = (0,0), (4,3), (8,6)$, so we need only subtract three marks, for a total of 13.

S89S3. The arithmetic mean of two numbers is never less than their geometric mean, so
$1/(\sqrt{1+x}) + 1/\sqrt{(1-x)} \geq 2/(\sqrt{(1+x)(1-x)}) = 2/(\sqrt{1-x^2})$. Now

$x^2 \geq 0$, so $1 - x^2 \leq 1$, and $1/\sqrt{(1-x^2)} \geq 1$.

Hence $2/\sqrt{(1-x^2)} \geq 2$, and the given expression is also at least 2. Equality holds (for the AM-GM inequality), if the two numbers are equal. This happens when $x = 0$, which gives the minimal value of 2.

S89S4. Drawing lines parallel to AB through points X and Y, it is not hard to see that the figure is dissected into six squares. This construction shows that AB:BC = 2:3.



S89S5. For all x, we have $f(1-x) = 1/\sqrt{1 - (1-x)} = 1/\sqrt{x}$. Since $f(a) = 2$, $\sqrt{1-a} = 1/2$, $1-a = 1/4$, and $a = 3/4$ Thus $f(1-a) = 2/\sqrt{3}$.

S89S6. We can obtain the figure described by "carving" it out of a cube. The piece missing turns out to be a pyramid, whose base is half of one of the faces of the cube, whose vertex is another vertex of the cube, and whose altitude is an edge of the cube.

The volume of this pyramid is $(1/2)(1/3)(1)(1) = 1/6$, and since the volume of the cube is 1, the volume of the polyhedron described is 5/6.

SOLUTIONS

S89S7. The first such number is 102, and the last is 996. They form an arithmetic progression with common difference 6. But how many of them are there? We can find out by setting 996 = 102 + 6(n-1), which shows that n = 150. The sum of the arithmetic progression is then (150/2)(102 + 996) = 82350.

S89S8. If we reflect the figure described in the diameter of the semicircle, we obtain another rectangle, inscribed this time in the full circle. The largest such rectangle, cut in half, will give the answer to the original problem. But this largest rectangle is certainly a square, with area 24x24/2 = 288, so the largest possible rectangle inscribed in the semicircle is 144.

A proof that a square is the largest rectangle inscribed in a given circle is left for the student.

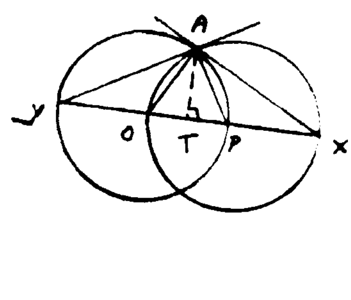
S89S9. If x > 1, the given inequality implies that $\log \sqrt{x} = (1/2)\log x > 1$, so that $\log x > 2$. This is true unless x < 100.

If x = 1, the implication above is not true, and by direct substitution, 1 does not satisfy the given inequality.

Hence the inequality is false for the integers 1 through 99, inclusive.

S89S10. Let a = 2x. Then x = a/2, and f(2x) = f(a) = $= (a/2)^2 - a/2 + 3$. This is an expression for the value of f at any point at all. Calling this point x, we have f(x) = $x^2/4 - x/2 + 3$.

S89S11. Since triangle AOP is equilateral, < APO = $60^\circ = \widehat{AO}$ (see diagram). Then <AXO = 30°. Also, <OAX = 90°, since a tangent is perpendicular to a radius at its point of contact. Hence <PXA = <PAX = <OAX -<OAP = $90^\circ - 60^\circ = 30^\circ$ = <OXA, so points O, P and X are collinear. Similarly, points O, P and Y are also collinear. Then XY = YO + OP + PX = 4+4+4 = 12. If the altitude to XY in triangle AXY is AT, we find (from 30-60-90 triangle AOT) that AT = $2\sqrt{3}$. Then the area of triangle AXY is $(1/2)(2\sqrt{3})(12) = 12\sqrt{3}$.



S89S12. For this arithmetic progression, the sum of n terms is given by $S_n = (n/2)(2\sqrt{11} + 44 - 4n)$. Since n is a positive integer, S_n is negative when the second factor above is negative. This condition leads to $\sqrt{11} + 22 - 2n < 0$, or $n > 11 + \sqrt{11}/2$. Since $3 < \sqrt{11} < 4$, we must choose n > 12. The smallest such value is n = 13.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER THREE SPRING, 1989

SOLUTIONS

S89S13. $2^y = 4 \cdot 8^x = 2^2 \cdot (2^3)^x = 2^{3x+2}$, so $y = 3x + 2$.

S89S14. One common proof that the area of a parallelogram is equal to a base times the corresponding altitude involves dissecting the parallelogram to form a rectange:

We can "imbed" this dissection in the situation of the problem, to find that each parallelogram is equal in area to a rectangle satisfying the same conditions. Thus the answer to this problem is given by problem S89S8. The area of the maximal parallogram is half the area of a square inscribed in the full circle. The area of such a square is here 200, so the largest possible parallelogram has area 100.

S89S15. If the polynomial has linear factors, then it must have rational roots. Because the coefficient of the lead term is 1, these roots must in fact be integral. Hence we are seeking to maximize the product of two numbers whose sum is -8. The maximum occurs when the numbers are equal, and its value is 16.

S89S16. We have

$$4 \cdot 4 \sin^2 x \cos^2 x = 3$$
$$4 \cdot (2 \sin x \cos x)^2 = 3$$
$$4 \sin^2 2x = 3$$
$$\sin^2 2x = 3/4$$
$$\sin 2x = \sqrt{3}/2$$
$$2x = 60^\circ$$
$$x = 30^\circ.$$

S89S17. The second term of a geometric progression is the geometric mean of the first and the third terms. Hence its square is $(\sqrt{2})(3\sqrt{2}) = 6$, so the second term is $\sqrt{6}$.

S89S18. Method I: $\dfrac{1}{p^2} + \dfrac{1}{q^2} = \dfrac{p^2 + q^2}{p^2q^2} = \dfrac{(p+q)^2 - 2pq}{p^2q^2} =$

$(1 - 18)/81 = -17/81$.

Method II: We can read the equation as giving a condition that certain numbers p and q must satisfy. From this, we can deduce a condition which the numbers $1/p^2$ and $1/q^2$ must satisfy. This will be another equation, the sum of whose roots is the number we require.

To accomplish this, we let $y = 1/x^2$, and change the equation to one involving y. We have $x = 1/\sqrt{y}$, so that

$$1/y - 1/\sqrt{y} + 9 = 0$$
$$1/\sqrt{y} = 1/y + 9$$
$$1/y = 1/y^2 + 18/y + 81,$$
$$y = 1 + 18y + 81y^2, \text{ or } 81y^2 + 17y + 1 = 0.$$

The sum of the roots of this last equation is, by inspection, -17/81.

For more on this method of "transformation of equations", see Hall and Knight, or any book on advanced algebra.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FOUR SPRING, 1989

SOLUTIONS

S89S19. It is not hard to see that the average of the first ten terms is the same as the average of the fifth and sixth term. Hence the sum is ten times the average, or $(10)(1/2)(m+n) = 5m + 5n$.

S89S20. The expression $\sin^4 a + \cos^4 a$ can be written as

$\sin^4 a + 2\sin^2 a\cos^2 a + \cos^4 a - 2\sin^2 a\cos^2 a$, which equals

$(\sin^2 a + \cos^2 a)^2 - 2\sin^2 a\cos^2 a = 1 - 2\sin^2 a\cos^2 a =$

$= 1 - (1/2)\sin^2 2a = 1 - 1/98 = 97/98$.

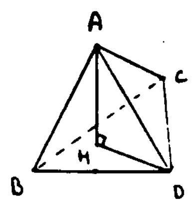
S89S21 If AX = XY = a, YB = b, then $2a + b = 6$, and we wish to minimize a^2b. To relate this product to the constant sum, we can use the fact that the geometric mean of three numbers is never less than their arithmetic mean:

$\sqrt[3]{a^2b} \leq (a+a+b)/3 = 2$, so $a^2b \leq 8$. Equality occurs when a = b, so that AX=XY=YB.

S89S22. We can calculate the volume of a tetrahedron by treating it as a pyramid. The volume of a pyramid is one-third the product of the height by the area of the base.

Each face of the regular tetrahedron is an equilateral triangle with side 12, altitude $6\sqrt{3}$, and therefore with area $36\sqrt{3}$.

To calculate the height of the tetrahedron, we can use right triangle AHD (see diagram). AD = 12, and DH is part of an altitude of face BCD. Since an altitude of an equilateral triangle is also a median, DH is, in fact, 2/3 the altitude of a face, or $(2/3)(6\sqrt{3}) = 4\sqrt{3}$. Finally, $AH^2 = AD^2 - DH^2 = 144 - 48 = 96$, so $AH = \sqrt{96} = 4\sqrt{6}$.



The volume of the tetrahedron, then, is $(1/3)(36\sqrt{3})(4\sqrt{6}) = 144\sqrt{2}$.

Note that triangle AHD is not a 30-60-90 triangle: angle ADH is less than 60 degrees.

S89S23. The sequence $\langle a_n\rangle$ is easy to work with directly.

For sequence $\langle b_n\rangle$, we find:

$b_1 = a_1 = 2$,
$b_2 = a_2 = 4$,
$b_3 = a_4 = 8$,
$b_4 = a_8 = 16$,
$b_5 = a_{16} = 32$,

and in general $b_n = 2^n$. Hence $b_{10} = 2^{10} = 1024$.

S89S24. Method I: Expand the given expression and use the coefficients of the given equation to evaluate the resulting elementary symmetric functions of the roots.

Method II: The three numbers $a+1$, $b+1$, $c+1$ are roots of the equation $f(x-1) = 0$, or $(x-1)^3 - 3(x-1) + 7 = 0$. The product of the roots is the opposite of the constant term, which is also

$$-f(0) = -((-1)^3 - 3(-1) + 7) = -(-1+3+7) = -9$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR A DIVISION CONTEST NUMBER FIVE SPRING, 1989

SOLUTIONS

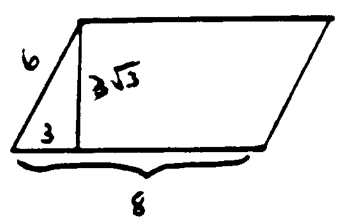
S89S25. Examining the remainders left by powers of 2 when divided by 17, we find that

2^0 has remainder 1
2^1 has remainder 2
2^2 has remainder 4
2^3 has remainder 8
2^4 has remainder -1 (i.e 16)
2^5 has remainder -2 (i.e 15)
2^6 has remainder -4 (i.e 13)
2^7 has remainder -8,(i.e. 9)

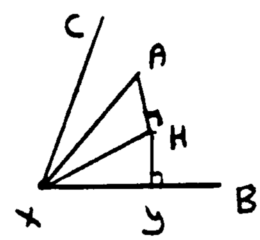
and that the pattern repeats thereafter. We must choose exactly three of these powers of 2, so that their remainders add up to zero. Since 0 itself is even, this means that there must be either 2 odd remainders (1 or -1) or no odd remainders. It is not difficult to see that the smallest N will result by choosing one of the remainders to be 1. It is then clear that a second remainder must also be 1, and the third must be -2.

This leads to the value $2^0 + 2^5 + 2^8$ for N, or $1 + 32 + 256 = 289$.

S89S26. The volume of a parallelopiped is equal to the product of the area of a face and the altitude to that face. We choose, for the face, a parallelogram with sides 6 and 8. The area of this parallelogram is $24\sqrt{3}$ (see diagram).



To find the altitude to this face, we can proceed as in problems S89S22. Let XA be an edge of length 3, XB an edge of length 8, and XC an edge of length 6. Draw AY⊥XB and draw AH perpendicular to plane CXB.

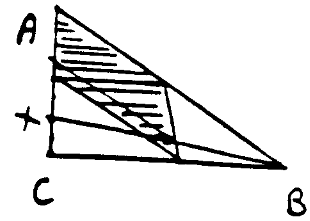


Then AXY is a 30-60-90 triangle, and $AY = 3\sqrt{3}/2$. By symmetry, angle HXY measures 30°, so that triangle HXY is again a 30-60-90 triangle, and $XH = 6/2\sqrt{3} = \sqrt{3}$. Then

$$AH^2 = AX^2 - XH^2 = 9 - 3 = 6, \text{ and } AH = \sqrt{6}.$$

The volume is $(24\sqrt{3})(\sqrt{6}) = 72\sqrt{2}$.

S89S27. To visualize region S, we first find the midpoints of all line segments with one endpoint at C and the other endpoint on AB. This can be obtained from AB by a dilation with factor 1/2 and center C. Thus this set is the line segment connecting the midpoints of AC and BC.



Next we pick a point X between A and C, and form the set of midpoints of segments with one endpoint at X and another on AB. In just the same fashion, this set is obtained from AB by a dilation with factor 1/2 and center X.

Thus region S is the union of line segments, each parallel to AB and equal to half of AB. It is now not hard to see that S is a parallelogram with one vertex at A and the other three at the midpoints of the sides of triangle ABC. Thus region S occupies half the area of the triangle, or 10 square units.

S89S28. Compare problem S89S20. The given expression can be written as

$$\frac{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - 1}{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) - 1}$$

or:

$$\frac{-2\sin^2 x\cos^2 x}{\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x - 1}$$

or:

$$\frac{-2\sin^2 x\cos^2 x}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - \sin^2 x\cos^2 x - 1}$$

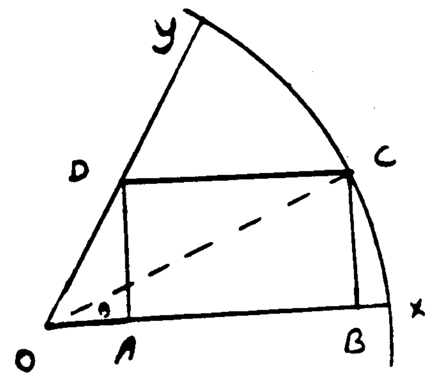
or:

$$\frac{-2\sin^2 x\cos^2 x}{-3\sin^2 x\cos^2 x}$$

or 2/3. The given expression is constant.

S89S29. We write the sum vertically, and examine the units column. Here there are 4! digits 5, since there are that many permutations of the remaining four digits. There are also just as many of each of the other digits. Therefore, the sum of the digits in the units column is 24(5+4+3+2+1) = 24 x 15 = 360.

Analysing each column of the sum separately, we find exactly the same result. Therefore, the total of the 120 numbers in the sum is 360 + 10x360 + 100x360 + 1000x360 + 10000x360 = = 360 x 11111 = 3999960.



S89S30. Let AD = a, CD = b. Then OA = a/2 (see diagram). Suppose angle COB has measure θ. Then a = sin θ, OB = cos θ, and b = OB-OA = cos θ -a/2 = = cos θ - (1/2)sin θ. It suffices to maximize the semi-perimeter of the rectangle. Let M = AB + AD. Then

$$M = a + b = (1/2)\sin\theta + \cos\theta.$$

To maximize M, we divide M by $\sqrt{5}/2$:

$$2M/\sqrt{5} = 1/\sqrt{5}\ \sin\theta + 2/\sqrt{5}\ \cos\theta.$$

Now $(1/\sqrt{5})^2 + (2/\sqrt{5})^2 = 1/5 + 4/5 = 1$, so there exists a number ϕ such that $0 < \phi < \pi/2$ and $\cos\phi = 1/\sqrt{5}$, $\sin\phi = 2/\sqrt{5}$. This means that

$$2M/\sqrt{5} = \cos\phi\sin\theta + \sin\phi\cos\theta = \sin(\phi + \theta).$$

Since the sine of an angle is at most 1, the maximal value of M is $\sqrt{5}/2$, and it occurs when θ and ϕ are complementary.

Let us see what this means for a and b. We have:

$$a = \sin\theta = \cos\phi = 1/\sqrt{5}$$
$$b = \cos\theta - a/2 = \sin\phi - a/2 = 2/\sqrt{5} - 1/2\sqrt{5} = 3/2\sqrt{5}.$$

Thus the ratio a:b is 1:(3/2) = 2:3.

The technique used on M works on any expression of the form $M = p\cos\theta + q\sin\theta$. We divide M by $\sqrt{p^2 + q^2}$, so that the coefficients of cos θ and sin θ are themselves the sine and cosine of some number. The technique can be visualized as maximizing the dot product of the vector (p,q) with the unit vector (cos θ, sin θ).

June 12, 1989

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1989 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	S89S1	885
	S89S6	5/6 or 2/3
	S89S9	100
Senior B	S89B4	This question was eliminated from the competition. If the rectangle is considered a square there are infinite solutions.

Have a great summer!

Sincerely yours,

Richard Geller

Secretary, NYCIML