NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER ONE SPRING, 1989

Part I: 10 Minutes

S89J1. Compute the product of the numbers

$$[(1988+2)(1988-2) + (1988-2)(1988+2)]$$

and

$$[(1988-2)(1988+2) - (1988+2)(1988-2)].$$

S89J2. In equilateral triangle ABC, point D is between B and C. If BD = 2, DC = 4, find AD.

Part II: 10 Minutes NYCIML CONTEST ONE SPRING, 1989

S89J3. If a and b are positive real numbers, and $1/a + 1/b = 1/6$, compute the least possible value of ab.

S89J4. Find the radius of a circle passing through the three vertices of a triangle with sides 5, 12, and 13.

Part III: 10 Minutes NYCIML CONTEST ONE SPRING, 1989

S89J5. Find the smallest positive integer with exactly nine (positive integral) divisors, including 1 and the integer itself.

S89J6. Zsa Zsa is drinking champagne from a glass in the shape of a right circular cone. The height of the cone is 9 cm., and the radius of the base is 4 cm. The glass is initially full to the brim. Zsa Zsa drinks exactly half the liquid,then puts the glass down on a level table. What is the height of the champagne in the glass?

ANSWERS

1. 0

2. $2\sqrt{7}$

3. 144

4. 6.5 or 13/2 or equivalent

5. 36

6. $\sqrt[3]{364.5}$ or $9/\sqrt[3]{2}$ or equivalent.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER TWO SPRING, 1989

Part I: 10 Minutes

S89J7. If $10^x = 5$, compute 10^{2x+1}.

S89J8. Point P is on hypotenuse AB of right triangle ABC, and is equidistant from legs AC and BC. If AP = 3 and PB = 4, compute the length of AC.

Part II: 10 Minutes NYCIML CONTEST TWO SPRING, 1989

S89J9. The positive integers p and p + 41 are both prime. Compute p.

S89J10. In parallelogram ABCD, points M and P trisect diagonal AC (with point M nearer A), and points N and Q trisect diagonal BD (with point N nearer B). If the area of parallelogram ABCD is 100, find the area of quadrilateral MNPQ.

Part III: 10 Minutes NYCIML CONTEST TWO SPRING, 1989

S89J11. In triangle ABC, points M and N trisect side AB (with M closer to A), and points P and Q trisect side AC (with P closer to A). Find the ratio of the area of trapezoid MPQN to the area of triangle ABC.

S89J12. The positive integer N is a multiple of 18, and has exactly ten positive divisors, including 1 and N itself. Compute N.

ANSWERS

7. 250

8. 21/5

9. 2

10. 100/9 or equivalent

11. 1:3 or equivalent

12. 162

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION CONTEST NUMBER THREE SPRING, 1989

Part I: 10 Minutes

S89J13. If x - 432 is 10 more than y + 568, compute x - y.

S89J14. Two sides of a triangle have lengths 3 and 6. The sum of the altitudes to these two sides is equal to twice the length of the altitude to the third side of the triangle. Compute the length of this third side.

Part II: 10 Minutes NYCIML CONTEST THREE SPRING, 1989

S89J15. The integers p, p + 28, and p + 56 are all positive prime numbers. Compute p.

S89J16. In isosceles trapezoid ABCD, diagonals AC, BD are perpendicular. If AB = 4 and AD = 7, compute the length of CD.

Part III: 10 Minutes NYCIML CONTEST THREE SPRING, 1989

S89J17. In trapezoid ABCD, base AB = 3 and base CD = 7. Points M and N are the midpoints of legs AD and BC respectively. Compute the ratio of the area of ABNM to that of MNCD.

S89J18. If a and b are integers, each greater than one, and $\sqrt{a\sqrt{a\sqrt{a}}} = b$, compute the smallest possible value of a + b.

ANSWERS

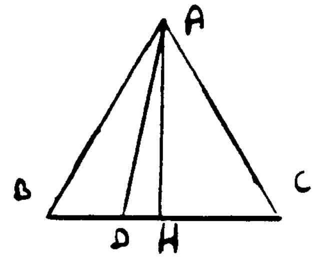
13. 1010	15. 3	17. 2:3 or equivalent
14. 4	16. $\sqrt{82}$	18. 384

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE SPRING, 1989

SOLUTIONS

S89J1. The second factor given is equal to zero, so the entire product is zero.

S89J2. Draw $AH \perp BC$. Since H is the midpoint of BC, BH = 3 and DH = 1, so AH $= 3\sqrt{3}$ (from 30-60-90 triangle ABH). Then, in right triangle ADH,



$AD^2 = AH^2 + DH^2 = 27 + 1 = 28$. Thus $AD = \sqrt{28} = 2\sqrt{7}$.

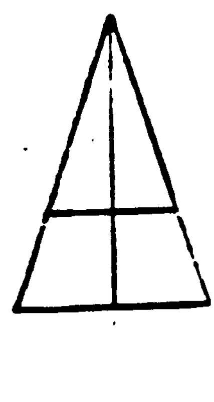
S89J3. Letting $1/a = x$, $1/b = y$, we have $x + y = 1/6$, and we must minimize $1/xy$.This is equivalent to maximizing the product xy. But if the sum of two numbers is constant, the maximum product occurs when the numbers are equal. This condition leads to $x = y = 1/12$, so that $a = b = 12$, and $ab = 144$.

S89J4. The triangle is a right triangle, so that the hypotenuse is a diameter of the circumscribed circle. Hence a radius of the circle must be $13/2 = 6.5$

S89J5. A number with oddly many divisors must be a perfect square (otherwise, pairs of distinct divisors can be formed, whose product is the given number). In fact, the number must be of the form $(pq)^2$, where p and q are distinct primes. The smallest such number is 36.

For more on the "number of divisors" function, usually written d(n), see any book on elementary number theory.

S89J6. The original volume is $\pi R^2H/3 = 48\pi$ cm.^{3}. The new volume is thus $24\pi cm^3$. If the new height is h, and the new radius r, then $\pi r^2h/3 = 24\pi$, and $r^2h = 72$. Since the cross sections of the volume of champagne remain similar (see diagram) $r:h = 4:9$,



and $r = 4h/9$. Then $16h^3/81 = 72$ and

$h^3 = 72(81)/16 = 364.5$

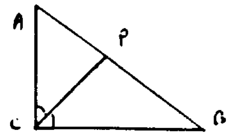
or $h^3 = 9(81)/2 = 9^3/2$. Therefore

$h = 9/\sqrt[3]{2}$ or $\sqrt[3]{364.5}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION   CONTEST NUMBER TWO   SPRING, 1989

SOLUTIONS

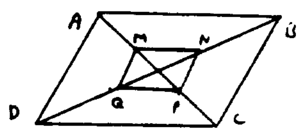
S89J7. $10^{2x+1} = 10^x \cdot 10^x \cdot 10 = 5 \times 5 \times 10 = 250$.



S89J8. It follows that point P is on the bisector of angle ACB, so that AP:PC = AC:BC = 3:4. Letting AC = 3x, BC = 4x, we can form an equation in x: $(3x)^2 + (4x)^2 = 49$. The solution is x = 7/5, so that AC = 21/5.

S89J9. If p is odd, then p + 41 is even, and if p is even, then p + 41 is odd. Thus one of these two prime numbers is even. Since the only even prime number is 2, one of them must be 2. Since p < p + 41, p = 2.

S89J10. If point X is the intersection of the diagonals, then MNPQ can be obtained from ABCD by dilating about point X by a factor of 1/3. Hence MNPQ is a parallelogram similar to the original, with 1/9 the area.



The figures ABCD and MNPQ are examples of <u>homothetic</u> figures. For more on this topic, see for instance Yaglom, <u>Geometric Transformations</u>.

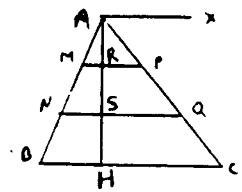
S89J11. Draw altitude AH and line AX || MP (see figure). Since AX, MP, NQ, BC divide AB equally, they also divide AH equally. Using absolute value for area, then, we have:

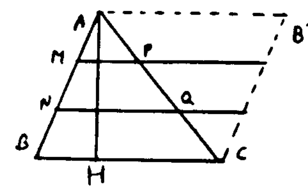
AR:AS = 1:2, and |AMP|:|ANQ| = 1:4,

AS:AH = 2:3, and |ANQ|:|ABC| = 4:9, (since the ratio of the areas of similar figures is the square of the ratio of their sides).

Thus |ANQ| = (4/9)|ABC|, and |MPQN| = (3/4)|ANQ| = (1/3)|ABC|.

Diagram II shows an alternative solution, in which the entire triangle is reflected in the midpoint of one of its sides to form a parallelogram. The ratios of the areas can then be deduced from the symmetry of the figure.





S89J12. The prime factorization of N looks like $2^a \cdot 3^b \cdot p^c \cdot q^d \cdot \ldots$, where a, b, c, d, ... are natural numbers, and p, q, ... are distinct primes greater than 3. Since 18 divides N, a must be at least 1 and b must be at least 2. Now the number of divisors of N is equal to $(a+1)(b+1)(c+1)\ldots$, which must equal 10. Since 10 factors only as 2x5 (the factorization 1x10 is not possible here), we must have a = 1, b = 4. Then $N = 2 \cdot 3^4 = 162$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER THREE — SPRING, 1989

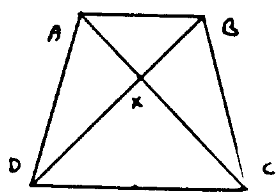
SOLUTIONS

S89J13. Subtracting, we find $(x-432) - (y+568) = 10$, or $(x-y) - 1000 = 10$, and $x - y = 1010$.

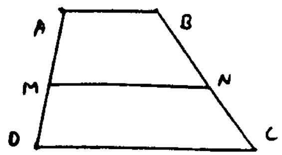
S89J14. If the sides of the triangle are a, b, and c (with $a = 3$ and $b = 6$), let h_a, h_b, h_c be the respective altitudes. Now $ah_a = bh_b = ch_c$, since each expression is twice the area of the triangle. This relationship can also be written $h_a/h_c = c/a$ and $h_b/h_c = c/b$. Then $h_a + h_b = 2h_c$, or $h_a/h_c + h_b/h_c = 2$, or $c/a + c/b = 2$. Substituting $a = 3$ and $b = 6$, we find that $c = 4$.

S89J15. Compare problem S89J9. Here, we can analyze the three numbers with respect to the prime 3, rather than the prime 2. Upon division by 3, there are three possible remainders: 0, 1, or 2. If p has remainder 1 upon division by 3 (that is, if it is congruent to 1 modulo 3), then $p-1$ is a multiple of 3, and $p + 28 = (p-1) + 27$ is also a multiple of 3--and hence is not prime. If p is congruent to 2 modulo 3, then $p-2$ is a multiple of 3, and $p + 56 = (p-2) + 54$ is a multiple of 3, and hence is not prime. Thus p must be a multiple of 3. The only such prime is 3 itself.

S89J16. If X is the point of intersection of the diagonals, it follows from the symmetry of the figure that AX = BX, so that AXB is an isosceles right triangle. Hence $AX = 2\sqrt{2}$. Then, in right triangle AXD, $AD^2 = AX^2 + XD^2$, so $XD = \sqrt{41}$. Since triangle DXC is also a right isosceles triangle, $CD = (\sqrt{41})(\sqrt{2}) = \sqrt{82}$.



S89J17. Each smaller quadrilateral is a trapezoid, and since AB, MN, CD are three parallels dividing AD equally, they divide any transversal equally. This the altitudes of the two smaller trapezoids are each half that of the larger trapezoid. If the large altitude is h, then, using absolute value for area:



$|ABNM| = (1/2)(h/2)(AB+NM) = (h/4)(3+5)$,

$|MNCD| = (1/2)(h/2)(MN+CD) = (h/4)(5+7)$, so the required ratio is $(3+5)/(5+7) = 8:12 = 2:3$.

S89J18. We have first $\sqrt{a\sqrt{a\sqrt{a}}} = \sqrt{\sqrt{a^3\sqrt{a}}} = \sqrt{\sqrt{\sqrt{a^7}}} = \sqrt[8]{a^7}$. Raising each side of the given equation to the eighth power gives $a^7 = b^8$. Now suppose p is a prime which divides b. Then p divides a as well. Suppose that the highest power of p which divides a is p^x, and the highest power of p which divides b is p^y. Then $a = p^x \cdot a'$, $b = p^y \cdot b'$, with a', b' both relatively prime to p. Then $a^7 = p^{7x} \cdot a'^7 = b^8 = p^{8y} \cdot b'^8$, so that $7x = 8y$. Thus x is a multiple of 8 and y is a multiple of 7, and a is at least p^8, which is at least 2^8 (since 2 is the smallest prime number). Similarly, b is at least 2^7. Setting $a = 256$, $b = 128$ gives a sum of 384 for $a + b$.